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THE GENERALIZED LANGEVIN EQUATION WITH POWER-LAW MEMORY IN A NONLINEAR POTENTIAL WELL

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Abstract. The generalized Langevin equation (GLE) is a stochastic integro-differential equation that has been used to describe the velocity of microparticles in viscoelastic fluids. In this work, we consider the large-time asymptotic properties of a Markovian approximation to the GLE in the presence of a wide class of external potential wells. The qualitative behavior of the GLE is largely determined by its memory kernel K, which summarizes the delayed response of the fluid medium on the particles past movement. When K can be expressed as a finite sum of exponentials, it has been shown that long-term time-averaged properties of the position and velocity do not depend on K at all. In certain applications, however, it is important to consider the GLE with a power law memory kernel. Using the fact that infinite sums of exponentials can have power law tails, we study the infinite-dimensional version of the Markovian GLE in a potential well. In the case where the memory kernel K is integrable (i.e. in the asymptotically diffusive regime), we are able to extend previous results and show that there is a unique stationary distribution for the GLE system and that the long-term statistics of the position and velocity do not depend on K. However, when K is not integrable (i.e. in the asymptotically subdiffusive regime), we are able to show the existence of an invariant probability measure but uniqueness remains an open question. In particular, the method of asymptotic coupling used in the integrable case to show uniqueness does not apply when K fails to be integrable.

Keywords: anomalous diffusion, asymptotic coupling, invariant measure.

1. Introduction

The movement of microparticles in biological fluids is often distinct from classical Brownian motion [19]. While some particles exhibit non-Gaussian [46, 31] and/or switching behavior [38, 41], an important category of *anomalous diffusion* includes paths with stationary Gaussian increments that are negatively correlated

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[24, 8, 47, 18]. In particular, anticorrelation can accumulate in such a way that the particle process $\{x(t)\}_{t\geq 0}$ has a mean squared displacement (MSD), $\mathbb{E}\,[x^2(t)]$, that is sublinear over a significant period of time [39, 29, 35]. Such a process is commonly called *subdiffusive*. There have been substantial efforts to model [33, 25, 36, 21, 13] and statistically analyze [24, 9, 6, 37, 32] individual, unconstrained, subdiffusive, microparticle paths. There have also been several studies concerned with how to incorporate external forces in models of viscoelastic diffusion. Originally, external forces were limited to movement in harmonic potentials [2, 29, 25], but more recently there has been success in analyzing nonlinear potentials [12, 43]. However, as we demonstrate below, there has been less progress studying fully subdiffusive behavior in nonlinear potential wells, which holds interest both in theory and in practical applications.

For theoreticians, there are several canonical questions associated with studying classical models like diffusion in double-well potentials [13]. For example, what impact does memory have on the long-term statistics for the position and velocity? How does memory affect the rate of transitions between the wells? In terms of applications, there are interesting emerging questions concerning the transport of intracellular cargo in the cytoplasm of cells. Organelles and other microparticles have been observed to exhibit significant subdiffusive behavior in cytoplasm [45, 49]. This poses a significant modeling challenge [14], especially in light of the observation that transport is commonly mediated through molecular motors that can act like non-Hookean springs [30]. Moreover, it is commonplace for experimentalists to probe fluid-mechanical properties of live cells by manipulation through "optical tweezers" [40, 48, 50]. However, such studies rarely take into account the subdiffusive character of the microparticle probes, or the likely nonlinear forces exerted by the trap.

To describe microparticle motion in *viscous* fluids, it is common to use a Langevin framework. Let $\{(x(t), v(t))\}_{t\geq 0}$ denote the position and velocity of a particle, and let $\Phi(x)$ denote the particle's potential energy at the position x. (Because it does not have a substantive impact on our results, we will study the dynamics in one dimension.) Newton's Second Law yields [43]

$$m\,dv(t) = -\gamma v(t) - \Phi'(x(t))dt + \sqrt{2\gamma}dW(t), \tag{1.1}$$

where $x'(t) = v(t)$. Here m is the particle's mass, γ is the viscous drag coefficient, and $W(t)$ is a standard Brownian motion. To be physically correct, the coefficient of the noise should be $\sqrt{2k_BT\gamma}$, where k_B is Boltzman's constant and T is the temperature, but for the sake of notational simplicity we will set $k_BT = 1$ throughout. Under appropriate conditions on the potential well, this system has a unique stationary distribution with density

$$\pi(x,v) \propto \exp\Big(-\big(\Phi(x)+\frac{m}{2}v^2\big)\Big) \tag{1.2}$$

and is geometrically ergodic, in the sense that the law of the process converges to the stationary distribution exponentially quickly (see, for example, [34, 5, 17, 43] as well as [4, 15] for related results). Birkoff's ergodic theorem in turn implies

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(x(s),v(s))ds=\int_{\mathbb{R}^2} f(x,v)\pi(x,v)dxdv, \quad \pi\text{-a.s. and in } L^1(\pi) \tag{1.3}$$

for any $f\in L^1(\pi)$. Ultimately, we are interested in whether or not such a property holds for viscoelastic diffusion in a nonlinear potential well.

In order to model the memory effects arising in viscoelastic diffusion, physicists have long employed the generalized Langevin equation (GLE), which adds a memory kernel $K:\mathbb{R}\to\mathbb{R}_+$ and a stationary Gaussian process $\{F(t)\}_{t\in\mathbb{R}}$ to the Langevin dynamics (1.1). In particular, the GLE in a potential well Φ has the form

$$m\,\dot v(t)=-\gamma v(t)-\Phi'(x(t))-\int_{-\infty}^t K(t-s)v(s)\,ds+F(t)+\sqrt{2\gamma}\,\dot W(t). \tag{1.4}$$

for $t\geq 0$, where we assume that $\mathbb{E}\,[F(t)F(s)]=K(t-s)$ in order to satisfy the fluctuation-dissipation relation [26]. In general, there can be a pair of coefficients in front of the memory terms, but they do not affect our analysis. We refer the reader to [20] for a physical interpretation of those parameters.

Due to the memory kernel K, in this formulation of GLE the process $(x(t),v(t))$ is non-Markovian. Therefore, it is not immediately clear what we mean by a "stationary distribution." However, if the memory kernel $K(t)$ is a sum of N exponential functions, we can use the so-called Mori-Zwanzig formalism [51, 9, 12, 42] to rewrite the GLE as a $2+N$ dimensional system of SDEs. When Φ is quadratic, such a representation is statistically equivalent to (1.4); otherwise, we simply call this the Markovian approximation of the GLE. This Markovian version of the GLE does admit a stationary distribution and one can show that the marginal distribution of the pair (x,v) in stationarity is exactly (1.2) [43]. The fact that the memory kernel does not affect the stationary statistics of x and v is, in some sense, a natural generalization of the observation that the drag coefficient γ does not appear in $\pi(x,v)$ for viscous diffusion. It is then reasonable to ask whether this property holds for general forms of K.

The sum-of-exponential form for K is a very useful construct, but it turns out that restricting ourselves to *finitely* many terms neglects an important qualitative regime. Indeed, if $K\in L^1(\mathbb{R})$, then the associated unconstrained GLE ($\Phi\equiv 0$) is always *asymptotically diffusive* in the sense that $\mathbb{E}\left[x^2(t)\right] \sim t$ as $t \to \infty$ [42, 35]. Here we write $f(t) \sim g(t)$ as $t \to \infty$ if $\lim_{t\to\infty} f(t)/g(t) = C \in (0, \infty)$. However, if $K \notin L^1(\mathbb{R})$ but $K(t) \sim t^{-\alpha}$ for some $\alpha \in (0,1)$ as $t \to \infty$, then under mild restrictions, the unconstrained GLE is *asymptotically subdiffusive*, i.e. $\mathbb{E}\left[x^2(t)\right] \sim t^\alpha$ [35]. We are primarily interested in memory kernels with power law tails, which can fall in either qualitative regime. As has been observed elsewhere [1, 12], it is possible for an *infinite* sum of exponentials to have a power law tail. Therefore, an infinite-dimensional version of the Mori-Zwanzig formalism is an appropriate way to study the GLE with power law memory.

In this work, we explore the infinite-dimensional Markovian version of the GLE with an eye toward addressing the fundamental question of whether $(x(t), v(t))$ is ergodic in the sense of (1.3). In Section 2 we introduce notation, explicitly define our model, and summarize the main results. In Section 3 we establish well-posedness for all values of $\alpha > 0$ (including both the asymptotically diffusive and subdiffusive cases). Using an extension of the invariant measure found by Pavliotis for the finite-dimensional GLE [43], in Section 4 we demonstrate the existence of an explicitly defined invariant probably measure in the infinite-dimensional case. In Section 5 we use *asymptotic coupling* [16] to establish uniqueness of this measure in the asymptotically diffusive case, but cannot extend the result to the asymptotically subdiffusive case. We have yet to determine whether this is a shortcoming of current methods, or if the claim is simply not true. We hope that this work will shine some light on this interesting question.

2. Notation and Rigorous Summary of Results

Suppose that a memory kernel $K(t)$ has the form

$$K(t) = \sum_{k=1}^{N} c_k e^{-\lambda_k t}, \tag{2.1}$$

where $c_k,\ \lambda_k > 0,\ k = 1, \ldots, N$. Then, following Chapter 8 of [43], we can use Duhamel's formula and set

$$z_k(t) = e^{-\lambda_k t} z_k(0) + \sqrt{c_k} \int_0^t e^{-\lambda_k (t-s)} v(s)\, ds + \sqrt{2\lambda_k} \int_0^t e^{-\lambda_k (t-s)}\, dW_k(s).$$

In order to approximate equation (1.4) as an $(N+2)$-dimensional Markov system, we write

$$\begin{aligned} dx(t) &= v(t)\,dt, \\ m\,dv(t) &= \Big(-\gamma v(t) - \Phi'(x(t)) - \sum_{k=1}^{N} \sqrt{c_i} z_i(t)\Big)dt + \sqrt{2\gamma}\,dW_0(t), \\ dz_k(t) &= (-\lambda_k z_k(t) + \sqrt{c_k} v(t))\,dt + \sqrt{2\lambda_k}\,dW_k(t), \qquad 1 \le k \le N, \end{aligned} \tag{2.2}$$

where $z_k(0)$ are i.i.d. $\mathcal{N}(0,1)$ random variables and $(W_0, W_1, \ldots, W_N)$ is a standard $(N+1)$-dimensional Brownian motion. This is the content of Proposition 8.1 of [43]. It is also known (see Proposition 8.2, [43]) that the system (2.2) is uniquely ergodic with an invariant probability density function $\varrho(x, v, z_1, \ldots, z_N)$ given by

$$\varrho(x, v, z_1, \ldots, z_N) \propto \exp\Big\{ -\Phi(x) - \frac{m}{2}v^2 - \frac{1}{2}\sum_{k=1}^{N} z_k^2 \Big\}. \tag{2.3}$$

As discussed above, in order to study power law memory kernels we consider infinite sums of exponential functions. To this end, let $\alpha, \beta > 0$ be given, and define constants c_k, λ_k, $k = 1, 2, \ldots$, by

$$c_k = \frac{1}{k^{1+\alpha\beta}}, \;\; \lambda_k = \frac{1}{k^\beta}. \tag{2.4}$$

Define the kernel K by

$$K(t) = \sum_{k\ge 1} c_k e^{-\lambda_k t}. \tag{2.5}$$

It follows (see Example 3.3 of [1]) with this choice of constants c_k and λ_k that

$$K(t) \sim t^{-\alpha} \quad \text{as} \quad t \to \infty, \tag{2.6}$$

thus giving the desired power law tail for the memory kernel K. With this definition for K, we consider the following infinite-dimensional system of stochastic differential equations

$$\begin{aligned} dx(t) &= v(t)\,dt, \\ m\,dv(t) &= \big(-\gamma v(t) - \Phi'(x(t)) - \sum_{k\ge 1} \sqrt{c_k} z_k(t)\big)\,dt + \sqrt{2\gamma}\,dW_0(t), \\ dz_k(t) &= (-\lambda_k z_k(t) + \sqrt{c_k} v(t))\;dt + \sqrt{2\lambda_k}\,dW_k(t), \qquad k \ge 1, \end{aligned} \tag{2.7}$$

where the W_k are independent, standard Brownian motions. Throughout this work, we will assume that the potential Φ satisfies the following growth and regularity conditions:

Assumption 1. *$\Phi \in C^\infty(\mathbb{R})$, $\int_{\mathbb{R}} |\Phi'| e^{-\Phi} dx < \infty$ and there exists a constant $b > 0$ such that for all $x \in \mathbb{R}$*

$$b(\Phi(x) + 1) \geq x^2.$$

A typical class of potentials Φ that satisfies Assumption 1 is the class of polynomials of even degree whose leading coefficient is positive.

Remark 2. *The first and third parts of Assumption 1 are quite standard, giving nominal regularity as well as assuring the potential grows at least as fast as a quadratic at infinity. The second condition is also a nominal growth condition on the derivative of Φ and will be used in Section 4 to check that our candidate invariant measure is indeed invariant.*

In order to define a phase space for the infinite-dimensional process

$$X(t) = (x(t), v(t), z_1(t), z_2(t), \dots),$$

we will make use of the Hilbert space $\mathcal{H}_{-s}$, $s \in \mathbb{R}$, equipped with the inner product $\langle \cdot, \cdot \rangle_{-s}$,

$$\mathcal{H}_{-s} = \Big\{ X = (x, v, z_1, z_2, \dots) : x^2 + v^2 + \sum_{k \geq 1} k^{-2s} z_k^2 < \infty \Big\}, \tag{2.8}$$

and

$$\langle X, \widetilde{X} \rangle_{-s} = x\widetilde{x} + v\widetilde{v} + \sum_{k \geq 1} k^{-2s} z_k \widetilde{z}_k. \tag{2.9}$$

We denote by $\| \cdot \|_{-s}$ the norm in $\mathcal{H}_{-s}$ given by

$$\|X\|_{-s}^2 = x^2 + v^2 + \sum_{k \geq 1} k^{-2s} z_k^2. \tag{2.10}$$

The canonical basis $\mathcal{D} = \{e_x, e_v, e_1, e_2, \dots\}$ in $\mathcal{H}_{-s}$ is then given by

$$\begin{aligned} e_x &= (1, 0, 0, 0, \dots), \\ e_v &= (0, 1, 0, 0, \dots), \\ e_k &= (0, 0, 0, \dots, k^s, 0, \dots),\ k \geq 1. \end{aligned} \tag{2.11}$$

From now on, for simplicity, we omit the subscript $-s$ in the inner product $\langle\cdot,\cdot\rangle$. In view of (2.11), for $X=(x,v,z_1,z_2,\dots)$, X we may write

$$X=\langle X,e_x\rangle e_x+\langle X,e_v\rangle e_v+\sum_{k\geq 1}\langle X,e_k\rangle e_k = xe_x+ve_v+\sum_{k\geq 1}k^{-s}z_ke_k. \tag{2.12}$$

Next, we collect several formulas involving Fréchet derivatives that will be useful later, especially in Section 4. For $\psi:\mathcal{H}_{-s}\to\mathbb{R}$, let $D\psi:\mathcal{H}_{-s}\to\mathcal{H}_{-s}$ be the first Fréchet derivative, if it exists. Then, the derivative of ψ in the direction of $e\in\mathcal{D}$ is given by

$$\langle D\varphi(X),e\rangle=\lim_{\varepsilon\to 0}\frac{\varphi(X+\varepsilon e)-\varphi(X)}{\varepsilon}=\frac{\partial\psi}{\partial\langle X,e\rangle}(X).$$

In view of representation (2.12), substituting e with e_x, e_v, e_k, $k\geq 1$ in the above formula gives

$$\langle D\psi(X),e_x\rangle=\frac{\partial\psi}{\partial x}(X),\qquad \langle D\psi(X),e_v\rangle=\frac{\partial\psi}{\partial v}(X), \tag{2.13}$$

$$\text{and } \langle D\psi(X),e_k\rangle=k^s\frac{\partial\psi\,(X)}{\partial z_k}$$

Similarly, if ψ is twice Fréchet differentiable, let $D^2\psi:\mathcal{H}_{-s}\to L(\mathcal{H}_{-s},\mathcal{H}_{-s})$ be the second Fréchet derivative, where $L(\mathcal{H}_{-s},\mathcal{H}_{-s})$ denotes the space of linear bounded maps from $\mathcal{H}_{-s}$ to itself. Then, for $e\in\mathcal{D}$, we have

$$\langle D^2\psi(X)(e),e\rangle=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\langle D\psi(X+\varepsilon e)-D\psi(X),e\rangle=\frac{\partial^2\psi}{\partial\langle X,e\rangle^2}(X).$$

Thus

$$\langle D^2\psi(X)(e_x),e_x\rangle=\frac{\partial^2\psi}{\partial x^2}\left(X\right),\qquad \langle D^2\psi(X)(e_v),e_v\rangle=\frac{\partial^2\psi}{\partial v^2}\left(X\right), \tag{2.14}$$

$$\text{and } \langle D^2\psi(X)(e_k),e_k\rangle=k^{2s}\frac{\partial^2\psi}{\partial z_k^2}\left(X\right).$$

Throughout, unless otherwise stated, we make the following assumptions about kernel parameters α,β as in (2.4) and (2.5) and the phase space regularity parameter s.

Assumption 3. *Let $\alpha,\beta>0$ be as in (2.4) and s as in (2.8). We assume that they satisfy either the* asymptotically diffusive *condition*

(D) $\alpha>1$, $\beta>\dfrac{1}{\alpha-1}$ *and* $\dfrac{1}{2}<s<\dfrac{(\alpha-1)\beta}{2}$;

or the critical *condition*

(C) $\alpha = 1$, $\beta > 1$ *and* $\dfrac{1}{2} < s < \dfrac{\beta}{2}$;

or the asymptotically subdiffusive *condition*

(SD) $0 < \alpha < 1$, $\beta > \dfrac{1}{\alpha}$ *and* $\dfrac{1}{2} < s < \dfrac{\alpha\beta}{2}$.

Remark 4. *The assumption above really concerns the parameters α, β only. Indeed, the particular choice of s in either part is the natural phase space range for the process for those particular choices of α, β. It is also worth remarking that, so long as $\beta > 0$ is large enough, the above simply splits the dynamics in the diffusive $(\alpha > 1)$ and the other two $(0 < \alpha < 1$, and $\alpha = 1)$ regimes.*

Remark 5. *In our context, to relate the infinite-dimensional system (2.7) to the original equation (1.4), the initial data $z_k(0)$ for (2.7) is necessarily i.i.d. with $\mathcal{N}(0,1)$ distribution. To ensure consistency, we want $(z_1(0), z_2(0), \dots)$ to live in $\mathcal{H}_{-s}$ almost surely. This means that we must restrict to the phase space $\mathcal{H}_{-s}$ for $s > 1/2$. It turns out that this constraint on s is also required to compute the density of the invariant measure, see Section 4.*

Fixing a stochastic basis $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, W)$ satisfying the usual conditions, cf. [23], where W is the cylidrical Wiener process defined later in (3.3), we state the following result giving existence and uniqueness of solutions of (2.7).

Proposition 6. *Suppose that Φ satisfies Assumption 1 and the constants α, β, s satisfy Assumption 3. Then for all initial conditions $X_0 \in \mathcal{H}_{-s}$, there exists a unique pathwise solution $X(\cdot, X_0) : \Omega \times [0,\infty) \to \mathcal{H}_{-s}$ of (2.7) in the following sense: $X(\cdot, X_0)$ is $\mathcal{F}_t$-adapted, $X(\cdot, X_0) \in C([0,\infty); \mathcal{H}_{-s})$ almost surely and that if $\widetilde{X}(\cdot, X_0)$ is another solution then for every $T \geq 0$,*

$$\mathbb{P}\left\{\forall t \in [0,T], X(t, X_0) = \widetilde{X}(t, X_0)\right\} = 1.$$

Moreover, for every $X_0 \in \mathcal{H}_{-s}$ and $T \geq 0$, there exists $C(T, X_0) > 0$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \|X(t)\|_{-s}^2\right] \leq C(T, X_0), \tag{2.15}$$

and that for any bounded set $B \subset \mathcal{H}_{-s}$, we have

$$\sup_{X_0\in B} C(T, X_0) < \infty. \tag{2.16}$$

The proof of Proposition 6 will be carried out in Section 3.

Our candidate stationary measure for the system (2.7) is an infinite-dimensional analogue of the one defined in (2.3). To write it down, let μ_x, μ_v and ν denote the probability measures on $\mathbb{R}$ defined by

$$\mu_x(dy) = \frac{1}{\int_{\mathbb{R}} e^{-\Phi(z)}\,dz} e^{-\Phi(y)}\,dy,\;\; \mu_v(dy) = \frac{\sqrt{m}}{\sqrt{2\pi}} e^{-\frac{my^2}{2}}\,dy, \qquad \text{and } \nu(dy) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dy. \tag{2.17}$$

Note that μ_x is indeed a probability measure on $\mathbb{R}$ by Assumption 1. We denote by μ the product probability measure on $\mathbb{R}^\infty$ given by

$$\mu = \mu_x \times \mu_v \times \prod_{k\geq 1} \nu. \tag{2.18}$$

Observe that since $s > 1/2$ by way of Assumption 3

$$\int_{\mathbb{R}^\infty} \|X\|_{-s}^2 \mu(dX) = \int_{\mathbb{R}^\infty} x^2 + v^2 + \sum_{k\geq 1} k^{-2s} z_k^2 \; \mu(dX) < \infty. \tag{2.19}$$

Thus the restriction of μ to $\mathcal{H}_{-s}$ is a probability measure as the above calculation shows that $\|X\|_{-s} < \infty$ μ-almost surely.

Let $X(t) = (x(t), v(t), z_1(t), z_2(t), \dots)$ be the solution of (2.7) and define the operator $\mathcal{P}(t) : \mathcal{B}_b(\mathcal{H}_{-s}) \to \mathcal{B}_b(\mathcal{H}_{-s})$ by

$$\mathcal{P}(t)\varphi(X_0) = \mathbb{E}\left[\varphi(X(t, X_0))\right]. \tag{2.20}$$

Here $\mathcal{B}_b(\mathcal{H}_{-s})$ is the space of bounded Borel measurable $\varphi : \mathcal{H}_{-s} \to \mathbb{R}$ and $C_b(\mathcal{H}_{-s})$ is the space of bounded continuous $\varphi : \mathcal{H}_{-s} \to \mathbb{R}$. It will be shown later in Proposition 15 that $\{\mathcal{P}(t)\}_{t\geq 0}$ is a Feller Markov semigroup on $C_b(\mathcal{H}_{-s})$. We recall that a finite measure ξ on $\mathcal{H}_{-s}$ is invariant for $\{\mathcal{P}(t)\}_{t\geq 0}$ if for every $\varphi \in C_b(\mathcal{H}_{-s})$ and $t > 0$

$$\int_{\mathcal{H}_{-s}} \mathcal{P}(t)\varphi(X)\xi(dX) = \int_{\mathcal{H}_{-s}} \varphi(X)\xi(dX).$$

We have the following result.

Theorem 7. *Suppose that Assumption 1 and Assumption 3 are satisfied. Then μ defined by (2.18) is an invariant probability measure for the Markov semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ on $C_b(\mathcal{H}_{-s})$ defined by (2.7).*

The idea behind the proof of Theorem 7 is simple but the details are non-trivial. This is because one tries to "integrate by parts" in infinite-dimensions aiming to show that $\mathcal{L}^*\mu = 0$ where $\mathcal{L}^*$ is some very formal adjoint of the Markov generator $\mathcal{L}$. The way that we circumnavigate this is by showing μ is "approximately invariant" for a sequence $X^R(t)$, $R \geq 1$, of processes which approximate $X(t)$ as $R \to \infty$. This turns out to be enough to show that μ is invariant for the original process.

Finally, our last result concerns unique ergodicity in the diffusive regime. To state it, we impose the following additional condition on the potential $\Phi(x)$.

Assumption 8. *There exist a function $f(x) : \mathbb{R} \to \mathbb{R}^+$ that is bounded on bounded sets and a positive number q such that for all $x, y \in \mathbb{R}$,*

$$|\Phi'(x) - \Phi'(y)| \leq |x - y| \left(f(x - y) + \Phi(x)^q \right). \tag{2.21}$$

Remark 9. *Assumption 8 is essentially a growth bound on the second derivative of Φ. In particular, if we assume further that there exist c, q_1, $q_2 > 0$ such that for all x, $y \in \mathbb{R}$ and $t \in [0, 1]$,*

$$\begin{aligned} &|\Phi''(x)| \leq c(\Phi(x)^{q_1} + 1), \\ \textit{and} \quad &\Phi((1-t)x + ty) \leq c(\Phi(x)^{q_2} + \Phi(|x - y|)^{q_2} + 1), \end{aligned} \tag{2.22}$$

then Φ satisfies Assumption 8. From this observation, it is a short exercise to see that Condition 2.22 includes not only the class of non-negative polynomials of even degree, but also functions that even grow faster than a polynomial at infinity, e.g. $\Phi(x) = e^{x^2}$.

Theorem 10. *Suppose Assumption 1, Assumption 8 and Condition (D) of Assumption 3 are satisfied. Then μ is the unique invariant measure for the Markov process defined by (2.7).*

The proof of Theorem 10, which will be given in Section 5, uses a asymptotic coupling argument following the ideas and results in the works of [10, 16, 27]. For a similar rigorous study of finite-dimesional Langevin Equation, we refer the reader to [34, 42].

3. Well-posedness

For notational convenience, throughout we write (2.7) more compactly as the following semilinear stochastic evolution equation

$$dX(t) = (LX(t) + F(X(t))\,dt + B\,dW(t), \qquad X(0) = X_0 \in \mathcal{H}_{-s}, \tag{3.1}$$

where L is a linear map given by

$$LX = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & -\frac{\gamma}{m} & -\frac{\sqrt{c_1}}{m} & -\frac{\sqrt{c_2}}{m} & \dots \\ 0 & \sqrt{c_1} & -\lambda_1 & 0 & \dots \\ 0 & \sqrt{c_2} & 0 & -\lambda_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x \\ v \\ z_1 \\ z_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} v \\ -\frac{\gamma}{m} v - \frac{1}{m} \sum_{k \geq 1} \sqrt{c_k} z_k \\ -\lambda_1 z_1 + \sqrt{c_1} v \\ -\lambda_2 z_2 + \sqrt{c_2} v \\ \vdots \end{pmatrix}, \tag{3.2}$$

where c_k, λ_k are as in (2.4) and the potential F is defined as

$$F(X) = (0, -\Phi'(x)/m, 0, 0, \dots)^T.$$

Regarding the stochastic term $B\,dW$, we may formally write

$$B\,dW(t) = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \frac{\sqrt{2\gamma}}{m} & 0 & 0 & \cdots \\ 0 & \sqrt{2\lambda_1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} dW_0 \\ dW_1 \\ dW_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\sqrt{2\gamma}}{m} dW_0(t) \\ \sqrt{2\lambda_1} dW_1(t) \\ \sqrt{2\lambda_2} dW_2(t) \\ \vdots \end{pmatrix}.$$

Following the formulation in, for example, [7], fix an auxiliary Hilbert space $\mathcal{W}$ and pick a complete orthonormal basis $\{e_k^{\mathcal{W}}\}_{k\geq 0}$. The cylindrical Wiener process $W(t)$ on the Hilbert space $\mathcal{W}$ is then defined as

$$W(t) = W_0(t)e_0^{\mathcal{W}} + W_1(t)e_1^{\mathcal{W}} + W_2(t)e_2^{\mathcal{W}} + \dots, \tag{3.3}$$

where the sequence $\{W_k(t)\}_{k\geq 0}$ are independent one-dimensional Brownian Motions. We can then define $B : \mathcal{W} \to \mathcal{H}_{-s}$ by its action

$$Be_0^{\mathcal{W}} = \frac{\sqrt{2\gamma}}{m} e_v, \qquad \text{and} \qquad Be_k^{\mathcal{W}} = \sqrt{2\lambda_k} k^{-s} e_k, \;\; k \geq 1, \tag{3.4}$$

where $\{e_x, e_v, e_1, e_2, \dots\}$ is the canonical basis of $\mathcal{H}_{-s}$, cf. (2.11). In view of (3.4), we have

$$BB^* e_x = 0, \qquad BB^* e_v = \frac{2\gamma}{m^2} e_v, \qquad \text{and} \qquad BB^* e_k = 2\lambda_k k^{-2s} e_k, \;\; k \geq 1. \tag{3.5}$$

In order to prove well-posedness of Equation (3.1), we need the following basic fact.

Proposition 11. *Suppose that* α, β *as in* (2.4) *and* s *as in* (2.8) *satisfy* $0 \leq 2s < \alpha\beta$. *Then,* $L : \mathcal{H}_{-s} \to \mathcal{H}_{-s}$ *defined in* (3.2) *is a bounded linear operator.*

Proof. Recalling (2.4), (2.10), (3.2) and invoking Cauchy–Schwarz inequality, we estimate

$$\begin{aligned}
&\|LX\|_{-s}^2\\
&= v^2 + \Big(-\frac{\gamma}{m}v - \frac{1}{m}\sum_{k\geq 1}\sqrt{c_k}z_k\Big)^2 + \sum_{k\geq 1}k^{-2s}(-\lambda_k z_k + \sqrt{c_k}v)^2\\
&\leq v^2 + \frac{2\gamma^2}{m^2}v^2 + \frac{2}{m^2}\Big(\sum_{k\geq 1}\sqrt{c_k}z_k\Big)^2 + \sum_{k\geq 1}2\lambda_k^2 k^{-2s}z_k^2 + \sum_{k\geq 1}2k^{-2s}c_k v^2\\
&\leq \Big(1 + \frac{2\gamma^2}{m^2} + 2\sum_{k\geq 1}k^{-2s}c_k\Big)v^2 + \frac{2}{m^2}\sum_{k\geq 1}c_k k^{2s}\sum_{k\geq 1}k^{-2s}z_k^2 + 2\sum_{k\geq 1}\lambda_k^2 k^{-2s}z_k^2\\
&= \Big(1 + \frac{2\gamma^2}{m^2} + 2\sum_{k\geq 1}k^{-2s}k^{-(1+\alpha\beta)}\Big)v^2 + \frac{2}{m^2}\sum_{k\geq 1}k^{-(1+\alpha\beta)}k^{2s}\sum_{k\geq 1}k^{-2s}z_k^2\\
&\qquad\qquad + 2\sum_{k\geq 1}k^{-\beta}k^{-2s}z_k^2\\
&\leq \Big(1 + \frac{2\gamma^2}{m^2} + 2\sum_{k\geq 1}k^{-(1+\alpha\beta+2s)} + \frac{2}{m^2}\sum_{k\geq 1}k^{-(1+\alpha\beta-2s)} + 2\Big)\|X\|_{-s}^2,
\end{aligned}$$

Since $\sum_{k\geq 1}k^{-(1+\alpha\beta-2s)}$ converges whenever $\alpha\beta > 2s$, the desired result follows. □

The proof of Proposition 6 follows a Lyapunov argument which we now explain. Let $\theta_R \in C^\infty(\mathbb{R};[0,1])$ satisfy

$$\theta_R(x) = \begin{cases} 1 & \text{if } |x| \leq R \\ 0 & \text{if } |x| \geq R+1 \end{cases} \tag{3.6}$$

and consider "cutoff" equation corresponding to (3.1)

$$dX^R(t) = \big[LX^R(t) + F\left(X^R(t)\right)\theta_R\left(x^R(t)\right)\big]\,dt + B\,dW(t), \qquad X^R(0) = X_0. \tag{3.7}$$

For each $R > 0$, it is not hard to prove that the global (in time) solution X^R exists and is unique, giving local (up until the time of explosion) pathwise existence and uniqueness of (3.1). Then, using a Lyapunov function $\Psi(X)$ that dominates the norm of X in $\mathcal{H}_{-s}$, we show a global bound on these solutions that does not depend on R, thereby obtaining global solutions of (3.1).

To prove Proposition 6, we begin with the following proposition.

Proposition 12 (Local Existence)**.** *Suppose that α, β as in (2.4) and s as in (2.8) satisfy $1 < 2s < \alpha\beta$. Let $X_0 \in \mathcal{H}_{-s}$ be given. For each $R > 0$, there exists a unique $X^R(t) \in L^2\left(\Omega, C\left([0,\infty); \mathcal{H}_{-s}\right)\right)$ satisfying (3.7).*

Remark 13. *We see that the conditions stated in Assumption 3 meet the hypothesis $1 < 2s < \alpha\beta$ of Proposition 12.*

Proof of Proposition 12. The linear map L is bounded by Lemma 11 and the nonlinear term in (3.7) is globally Lipschitz in $\|\cdot\|_{-s}$ by construction. Moreover, the additive noise term lives in $\mathcal{H}_{-s}$ almost surely since

$$\begin{aligned}\mathbb{E}\left\|\int_0^T B\,dW(t)\right\|_{-s}^2 &= \mathbb{E}\left|\int_0^T \frac{\sqrt{2\gamma}}{m} dW_0(t)\right|^2 + \sum_{k\ge 1} k^{-2s}\mathbb{E}\left|\int_0^T \sqrt{2\lambda_k}\,dW_k(t)\right|^2 \\ &= \frac{2\gamma T}{m^2} + 2T\sum_{k\ge 1} k^{-2s}\lambda_k < \infty.\end{aligned}$$

The corresponding solution hence exists and is unique by a standard Banach fixed point argument. □

Next, inspired by [34, 43], we introduce a Lyapunov function $\Psi : \mathcal{H}_{-s} \to [0,\infty)$ given by

$$\Psi(X) := \frac{1}{m}\Phi(x) + \frac{1}{2}v^2 + \frac{1}{2}\sum_{k\ge 1} k^{-2s} z_k^2. \tag{3.8}$$

Define $\mathcal{L} : C^2(\mathcal{H}_{-s}) \to \mathbb{R}$ to be the operator given by

$$\mathcal{L}\varphi(X) := \langle D\varphi(X), LX + F(X)\rangle + \frac{1}{2}\mathrm{Tr}(D^2\varphi BB^*).$$

In view of (2.13), (2.14), (3.2) and (3.5), $\mathcal{L}$ can be explicitly written as

$$\begin{aligned}\mathcal{L}\varphi(X) &= v\frac{\partial\varphi(X)}{\partial x} + \Big(-\frac{\gamma}{m}v - \frac{1}{m}\Phi'(x) - \frac{1}{m}\sum_{k\ge 1}\sqrt{c_k}z_k\Big)\frac{\partial\varphi(X)}{\partial v} \\ &\quad + \sum_{k\ge 1}\left(-\lambda_k z_k + \sqrt{c_k}v\right)\frac{\partial\varphi(X)}{\partial z_k} + \frac{\gamma}{m^2}\frac{\partial^2\varphi(X)}{\partial v^2} + \sum_{k\ge 1}\lambda_k\frac{\partial^2\varphi(X)}{\partial z_k^2}\end{aligned} \tag{3.9}$$

where $\varphi \in C^2(\mathcal{H}_{-s})$. Note that, once we establish Proposition 6, $\mathcal{L}$ is in fact the infinitesimal generator of the Markov semigroup $\{\mathcal{P}(t)\}_{t\ge 0}$ associated with (2.7). We assert the following proposition.

Proposition 14 (Global bound)**.** *Suppose that Assumption 1 and Assumption 3 are satisfied. Let $\Psi(X)$ be defined in (3.8) and $\mathcal{L}$ be the operator as in (3.9). Then, for every $X \in \mathcal{H}_{-s}$,*

$$\mathcal{L}\Psi(X) \leq a_1\Psi(X) + a_2, \tag{3.10}$$

where a_1, a_2 are finite constants that can be explicitly given as

$$a_1 = \max\Big\{1 + \frac{1}{m}, \frac{1}{m}\sum_{k\geq 1} c_k k^{2s} + \sum_{k\geq 1} c_k k^{-2s}\Big\} \text{ and } a_2 = \frac{\gamma}{m^2} + \sum_{k\geq 1} k^{-2s}\lambda_k.$$

Proof. Applying Assumption 3, first note that a_1 and a_2 are both finite since

$$\sum_{k\geq 1} c_k k^{2s} = \sum_{k\geq 1} \frac{1}{k^{1+\alpha\beta-2s}} < \infty \quad \text{and} \quad \sum_{k\geq 1} k^{-2s}\lambda_k = \sum_{k\geq 1}\frac{1}{k^{\beta+2s}} < \infty.$$

We now apply $\mathcal{L}$ to Ψ to see that

$$\begin{aligned}\mathcal{L}\Psi(X) = &-\frac{\gamma}{m}v^2 - \sum_{k\geq 1}\lambda_k k^{-2s} z_k^2 \\ &- \frac{1}{m}\sum_{k\geq 1}\sqrt{c_k} z_k v + \sum_{k\geq 1}\sqrt{c_k} k^{-2s} z_k v + \frac{\gamma}{m^2} + \sum_{k\geq 1} k^{-2s}\lambda_k.\end{aligned} \tag{3.11}$$

The cross terms between z_k and v can be bounded using Hölder's inequality as follows:

$$\begin{aligned}&-\frac{1}{m}\sum_{k\geq 1}\sqrt{c_k} z_k v + \sum_{k\geq 1}\sqrt{c_k} k^{-2s} z_k v \\ &\leq \frac{1}{2m}\Big(\sum_{k\geq 1} c_k k^{2s}\sum_{k\geq 1} k^{-2s} z_k^2 + v^2\Big) + \frac{1}{2}\Big(\sum_{k\geq 1} c_k k^{-2s}\sum_{k\geq 1} k^{-2s} z_k^2 + v^2\Big) \\ &= \frac{1}{2}\Big(1 + \frac{1}{m}\Big)v^2 + \frac{1}{2}\Big(\frac{1}{m}\sum_{k\geq 1} c_k k^{2s} + \sum_{k\geq 1} c_k k^{-2s}\Big)\sum_{k\geq 1} k^{-2s} z_k^2 \\ &\leq a_1\Psi(X),\end{aligned} \tag{3.12}$$

where $a_1 > 0$ is as in the statement of the result. We finally combine (3.11) with (3.12) to obtain (3.10). □

We are now ready to prove the main existence and uniqueness result for equation (3.1). The argument is classical and can be found in literature, e.g., [3, 11, 22].

Proof of Proposition 6. For every $R > 0$, let $X^R(t)$ be the unique solution of the cutoff system (3.7) given to us by Proposition 12. Define the stopping time

$$\tau_R = \inf\{t > 0 : \|X(t)\|_{-s} > R\}.$$

Note that, for all times $t < \tau_R$, X^R solves (3.1). Consequently, the solution (3.1) exists and is unique up until the *time of explosion* $\tau_\infty = \lim_{R\to\infty}\tau_R$, which is possibly finite on a set of positive probability. We show using the Lyapunov function Ψ above, cf. (3.8) that $\tau_\infty = \infty$ a.s.

By Ito's Formula we have that

$$\begin{aligned} d\Psi(X(t\wedge\tau_R)) = \mathcal{L}\Psi(X(t\wedge\tau_R))dt + \frac{\sqrt{2\gamma}}{m}v(t\wedge\tau_R)dW_0(t) \\ + \sum_{k\ge1}\sqrt{2\lambda_k}k^{-2s}z_k(t\wedge\tau_R)dW_k(t), \end{aligned} \tag{3.13}$$

where $\mathcal{L}$ is the operator defined in (3.9). We then infer the following bound

$$\begin{aligned} \mathbb{E}\left[\sup_{0\le t\le T}\Psi(X(t\wedge\tau_R))\right] \le \Psi(X_0) + \mathbb{E}\sup_{0\le t\le T}\int_0^t \mathcal{L}\Psi(X(r\wedge\tau_R))dr \\ + \mathbb{E}\sup_{0\le t\le T}\left|\int_0^t \frac{\sqrt{2\gamma}}{m}v(r\wedge\tau_R)dW_0(r) + \sum_{k\ge1}\sqrt{2\lambda_k}k^{-2s}z_k(t\wedge\tau_R)dW_k(r)dr\right|. \end{aligned} \tag{3.14}$$

Applying Proposition 14 on $\mathcal{L}\Psi(X(r\wedge\tau_R))$ and the Burkholder-Davis-Gundy inequality on the martingale term on the above RHS yields

$$\begin{aligned} \mathbb{E}\left[\sup_{0\le t\le T}\Psi(X(t\wedge\tau_R))\right] \le \Psi(X_0) + a_2T + a_1\mathbb{E}\int_0^T \sup_{0\le r\le t}\Psi(X(r\wedge\tau_R))dt \\ + c\left[\mathbb{E}\int_0^T \frac{2\gamma}{m^2}v(t\wedge\tau_R)^2 + \sum_{k\ge1}2\lambda_k k^{-4s}z_k(t\wedge\tau_R)^2dt\right]^{1/2}, \end{aligned} \tag{3.15}$$

where a_1, a_2 are as in (3.10) and $c > 0$ is the constant from Burkholder-Davis-Gundy's inequality and is independent of R, T, X_0. We observe now that the last integrand in (3.15) is dominated by $\Psi(X(t\wedge\tau_R))$. We thus infer that

$$\mathbb{E}\left[\sup_{0\le t\le T}\Psi(X(t\wedge\tau_R))\right] \le \Psi(X_0) + c_1 + c_2\int_0^T \mathbb{E}\left[\sup_{0\le r\le t}\Psi(X(r\wedge\tau_R))\right]dt, \tag{3.16}$$

where c_1, $c_2 > 0$ are constants independent of R and X_0. Gronwall's inequality then implies that

$$\mathbb{E}\left[\sup_{0\le t\le T}\Psi(X(t\wedge\tau_R))\right]\le\left(\Psi(X_0)+c_1\right)e^{c_2T}. \tag{3.17}$$

Also note that there exists a constant $c>0$ such that for all $X\in\mathcal{H}_{-s}$, $c(\Psi(X)+1)\ge\|X\|^2_{-s}$. We thus infer the existence of a constant $C(T,X_0)>0$ such that

$$\mathbb{E}\Big[\sup_{0\le t\le T}\|X(t\wedge\tau_R)\|^2_{-s}\Big]\le C(T,X_0). \tag{3.18}$$

Sending R to infinity in the above, we obtain by Fatou's lemma

$$\mathbb{E}\Big[\sup_{0\le t\le T}\|X(t\wedge\tau_\infty)\|^2_{-s}\Big]\le C(T,X_0). \tag{3.19}$$

Hence, this implies $\mathbb{P}\left\{T<\tau_\infty\right\}=1$ for any $T>0$. By sending T to infinity, we see that $\mathbb{P}\left\{\tau_\infty=\infty\right\}=1$, which implies the well-posedness of the global solution for any fixed $X_0\in\mathcal{H}_{-s}$.

We finally note that $C(T,X_0)$ is actually dominated by $\Psi(X_0)e^{c_2T}$ following from estimates (3.17) and (3.18). It is also clear that $\Psi(X)$ is bounded on bounded sets in $\mathcal{H}_{-s}$. We therefore obtain the bound in (2.16), which concludes the proof. □

In addition to pathwise existence and uniqueness of the solution of equation (3.1), we will need the following basic properties of the Markov semigroup $\mathcal{P}(t):\mathcal{B}_b(\mathcal{H}_{-s})\to\mathcal{B}_b(\mathcal{H}_{-s})$. We recall that $\mathcal{P}(t)$ defined as in (2.20) possesses the Markov property; namely, for every $X\in\mathcal{H}_{-s}$, $\varphi\in C_b(\mathcal{H}_{-s})$, t, $r\ge0$,

$$\mathcal{P}(t+r)\varphi(X)=\mathcal{P}(t)\left(\mathcal{P}(r)\varphi\right)(X).$$

Proposition 15. *Under the Hypothesis of Proposition 6, let $X(t)$ be the unique strong solution of (2.7) and $\mathcal{P}(t)$ be the corresponding Markov semigroup. We have the following:*

(a) *Whenever $X_k\to X_0$ in $\mathcal{H}_{-s}$*

$$\lim_{k\to\infty}\mathbb{E}\|X(t,X_k)-X(t,X_0)\|_{-s}=0. \tag{3.20}$$

(b) *$\mathcal{P}(t)$ has the Feller property: $\mathcal{P}(t)\varphi\in C_b(\mathcal{H}_{-s})$ whenever $\varphi\in C_b(\mathcal{H}_{-s})$.*

Proof. (a) For notational simplicity, throughout this proof, we shall omit the subscript $-s$ in the norm $\|\cdot\|_{-s}$. Denote by $X_{(k)}(t)$ the solution of (2.7) with initial data $X_{(k)}(0)=X_k$. Fixing $R>0$ to be chosen later, define the stopping time

$$\tau^k_R=\inf_{t\ge0}\{\|X_{(k)}(t)\|+\|X_{(0)}(t)\|\ge R\},$$

and observe that by Chebychev's inequality,

$$\begin{aligned}\mathbb{P}\left\{\tau_R^k \le t\right\} &= \mathbb{P}\left\{\sup_{0\le \ell\le t}\|X_{(k)}(\ell)\| + \|X_{(0)}(\ell)\| \ge R\right\}\\ &\le \frac{\mathbb{E}\left[\sup_{0\le\ell\le t}\|X_{(k)}(\ell)\| + \|X_{(0)}(\ell)\right]}{R}\\ &\le \frac{\sqrt{C(t,X_k)}+\sqrt{C(t,X_0)}}{R},\end{aligned}$$

where we used (2.15) in the last inequality. It follows from (2.16) that

$$\sup_k \mathbb{P}\left\{\tau_R^k \le t\right\} \le \frac{C(t)}{R}, \tag{3.21}$$

for a finite constant $C(t) > 0$ independent of R. Next, let $X^R_{(k)}$ and $X^R_{(0)}$ be the local solutions of (3.7) from Proposition 12. Since the drift term of (3.7) is Lipschitz, there exists a constant $c(R,t) > 0$ such that (see Theorem 9.1, [7]).

$$\mathbb{E}\left\|X^R_{(k)}(t) - X^R_{(0)}(t)\right\| \le C(R,t)\left\|X_k - X_0\right\|. \tag{3.22}$$

Now we have a chain of implications

$$\begin{aligned}&\mathbb{E}\|X_{(k)}(t)-X_{(0)}(t)\|\\ &\quad\le \mathbb{E}\left[\left(\|X_{(k)}(t)\|+\|X_{(0)}(t)\|\right)1_{\{\tau_R^k\le t\}}\right] + \mathbb{E}\left[\|X_{(k)}(t)-X_{(0)}(t)\|1_{\{\tau_R^k>t\}}\right]\\ &\quad= \mathbb{E}\left[\left(\|X_{(k)}(t)\|+\|X_{(0)}(t)\|\right)1_{\{\tau_R^k\le t\}}\right] + \mathbb{E}\left[\|X^R_{(k)}(t)-X^R_{(0)}(t)\|1_{\{\tau_R^k>t\}}\right]\\ &\quad\le \left(\mathbb{E}\left[\left(\|X_{(k)}(t)\|+\|X_{(0)}(t)\|\right)^2\right]\right)^{1/2}\left(\mathbb{P}\left\{\tau_R^k\le t\right\}\right)^{1/2} + \mathbb{E}\|X^R_{(k)}(t)-X^R_{(0)}(t)\|\\ &\quad\le \left(\mathbb{E}\left[\left(\|X_{(k)}(t)\|+\|X_{(0)}(t)\|\right)^2\right]\right)^{1/2}\left(\mathbb{P}\left\{\tau_R^k\le t\right\}\right)^{1/2} + C(R,t)\|X_k-X_0\|\\ &\quad\le \frac{C_1(t)}{R^{1/2}} + C(R,t)\left\|X_k-X_0\right\|,\end{aligned}$$

where note carefully that $C_1(t)$ is independent of k and R and $C(R,t)$ is independent of k. The above RHS now tends to zero by taking R sufficiently large first and then letting X_k sufficiently close to X_0. This establishes (a).

To prove (b), let $X_k \to X_0$ and $\varphi \in C_b(\mathcal{H}_{-s})$, we have to show $\mathcal{P}(t)\varphi(X_k) \to \mathcal{P}(t)\varphi(X_0)$. It suffices to show that for every subsequence $\{k_i\}$, there exists a further subsequence $\{k_{i_j}\}$ such that $\mathcal{P}(t)\varphi(X_{k_{i_j}}) \to \mathcal{P}(t)\varphi(X_0)$. In view of part (a), the sequence $X_{(k_i)}(t)$ converges to $X_{(0)}(t)$ in $L^1(\Omega;\mathcal{H}_{-s})$. We thus can extract a subsequence $X_{(k_{i_j})}(t)$ converging to $X_{(0)}(t)$ a.s. Since φ is bounded, applying the Dominated Convergence Theorem yields

$$\mathbb{E}\left[\varphi(X_{(k_{i_j})}(t))\right] \to \mathbb{E}\left[\varphi(X_{(0)}(t))\right] \text{ as } j \to \infty,$$

which implies (b) and thus completes the proof. □

4. Invariance of μ

In this section, we show that μ defined in (2.18) is invariant for the Markov semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$. We first sketch briefly the structure of the proof before diving into details.

The goal is to show that for every $\varphi \in C_b(\mathcal{H}_{-s})$ and $t \geq 0$ we have

$$\int_{\mathcal{H}_{-s}} \mathcal{P}(t)\varphi(X)\mu(dX) = \int_{\mathcal{H}_{-s}} \varphi(X)\mu(dX). \tag{4.1}$$

Let $C_b^2(\mathcal{H}_{-s})$ denote the space of real-valued functions on $\mathcal{H}_{-s}$ that have bounded first and second Fréchet derivatives. Approximating φ by functions in $C_b^2(\mathcal{H}_{-s})$ if necessary, it thus suffices to show that (4.1) holds for any $\psi \in C_b^2(\mathcal{H}_{-s})$

$$\int_{\mathcal{H}_{-s}} \mathcal{P}(t)\psi(X)\mu(dX) = \int_{\mathcal{H}_{-s}} \psi(X)\mu(dX). \tag{4.2}$$

In order to show (4.2), it is helpful to make use of the cutoff system (3.7) and the semigroup $\mathcal{P}^R(t)$ where for $R > 0$, $\mathcal{P}^R(t)$ is defined analogously to (2.20) by replacing $X(t)$ with $X^R(t)$ solving (3.7). The advantage of using the cutoff systems is that, because they have globally Lipschitz coefficients, they immediately satisfy the Kolmogorov backward equation, cf. Theorem 9.23 of [7]. This fact we will need later in the proof of Proposition 17. Specifically, we will prove that μ is *almost invariant* for the cutoff semigroup $\mathcal{P}^R(t)$; namely,

$$\int_{\mathcal{H}_{-s}} \mathcal{P}^R(t)\psi(X)\mu(dX) = \int_{\mathcal{H}_{-s}} \psi(X)\mu(dX) + \varepsilon^R(\psi, t), \tag{4.3}$$

where $\varepsilon^R(\psi, t)$ is a remainder term that (possibly) depends on ψ and t, and satisfies $\varepsilon^R(\psi, t) \to 0$ as $R \to \infty$. We will see that this then implies the desired equality (4.2).

Before proving Theorem 7, we collect several properties about Gaussian measures on $\mathbb{R}$ which follow simply by using integration by parts. Let μ_v, ν be as in (2.17). Then, for every $\varrho_1 \in C_b^2(\mathbb{R})$, it holds that

$$\int_{\mathbb{R}} -y\varrho_1'(y) + \frac{1}{m}\varrho_1''(y)\,\mu_v(dy) = 0, \tag{4.4}$$

and

$$\int_{\mathbb{R}} -y\varrho_1'(y) + \varrho_1''(y)\,\nu(dy) = 0. \tag{4.5}$$

Also, for every $\varrho_2 \in C_b^1(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2} \left(\frac{1}{m} z\partial_y \varrho_2(y,z) - y\partial_z \varrho_2(y,z)\right) (\mu_v \times \nu)\,(dy,dz) = 0. \tag{4.6}$$

With these observations, we have the following result:

Lemma 16. *Given $R > 0$, let $\mathcal{L}^R$ be the infinitesimal generator of the Markov semigroup $\{\mathcal{P}^R(t)\}_{t\geq 0}$ associated with X^R solving (3.7) and let μ be as in (2.18). Then, for every $\psi \in C_b^2(\mathcal{H}_{-s})$, we have the following equality*

$$\int_{\mathcal{H}_{-s}} \mathcal{L}^R\psi(X)\mu(dX) = \int_{\mathcal{H}_{-s}} v\Phi'(x)\left(1-\theta^R(x)\right)\psi(X)\mu(dX). \tag{4.7}$$

Proof. Similar to (3.9), $\mathcal{L}^R\psi$ is given by

$$\begin{aligned} \mathcal{L}^R\psi(X) = v\frac{\partial\psi(X)}{\partial x} &+ \left(-\frac{\gamma}{m}v - \frac{1}{m}\Phi'(x)\theta^R(x) - \frac{1}{m}\sum_{k\geq 1}\sqrt{c_k}z_k\right)\frac{\partial\psi(X)}{\partial v} \\ &+ \sum_{k\geq 1}\left(-\lambda_k z_k + \sqrt{c_k}v\right)\frac{\partial\psi(X)}{\partial z_k} + \frac{\gamma}{m^2}\frac{\partial^2\psi(X)}{\partial v^2} + \sum_{k\geq 1}\lambda_k\frac{\partial^2\psi(X)}{\partial z_k^2}. \end{aligned} \tag{4.8}$$

We integrate both sides against μ in $\mathcal{H}_{-s}$ and rearrange the above RHS appropriately to obtain

$$\begin{aligned} \int_{\mathcal{H}_{-s}} \mathcal{L}^R\psi(X)\mu(dx) &= \int_{\mathcal{H}_{-s}} \left[v\frac{\partial\psi(X)}{\partial x} - \frac{1}{m}\Phi'(x)\theta^R(x)\frac{\partial\psi(X)}{\partial v}\right]\mu(dX) \\ &\quad + \int_{\mathcal{H}_{-s}} \left[-\frac{\gamma}{m}v\frac{\partial\psi(X)}{\partial v} + \frac{\gamma}{m^2}\frac{\partial^2\psi(X)}{\partial v^2}\right]\mu(dX) \\ &\quad + \sum_{k\geq 1}\int_{\mathcal{H}_{-s}} \left[-\frac{\sqrt{c_k}}{m}z_k\frac{\partial\psi(X)}{\partial v} + \sqrt{c_k}v\frac{\partial\psi(X)}{\partial z_k}\right]\mu(dX) \\ &\quad + \sum_{k\geq 1}\int_{\mathcal{H}_{-s}} \left[-\lambda_k z_k\frac{\partial\psi(X)}{\partial z_k} + \lambda_k\frac{\partial^2\psi(X)}{\partial z_k^2}\right]\mu(dX) \\ &= I_{0,1} + I_{0,2} + \sum_{k\geq 1} I_{k,1} + \sum_{k\geq 1} I_{k,2}. \end{aligned} \tag{4.9}$$

At this point (4.9) is still formal. We need to show that $\mathcal{L}^R\psi \in L^1(\mathcal{H}_{-s},\mu)$ and that the above rearrangement is possible. To this end, we claim that the RHS after the first equality of (4.9) is absolutely convergent. Since $\psi \in C_b^2(\mathcal{H}_{-s})$, (2.13) and Parseval's identity imply a bound on first partial derivatives

$$\begin{aligned}\Big|\frac{\partial\psi(X)}{\partial x}\Big|^2+\Big|\frac{\partial\psi(X)}{\partial v}\Big|^2+\sum_{k\geq 1}k^{2s}\Big|\frac{\partial\psi(X)}{\partial z_k}\Big|^2&\\ &=\langle D\psi(X),e_x\rangle^2+\langle D\psi(X),e_v\rangle^2+\sum_{k\geq1}\langle D\psi(X),e_k\rangle^2\\ &=\|D\psi(X)\|_{-s}^2\\ &\leq \|D\psi\|_\infty^2,\end{aligned}\tag{4.10}$$

where for $D\psi:\mathcal{H}_{-s}\to\mathcal{H}_{-s}$, $\|D\psi\|_\infty=\sup_{Y\in\mathcal{H}_{-s}}\|D\psi(Y)\|_{-s}$. Similarly, (2.14) implies bounds on second partial derivatives,

$$\frac{\partial^2\psi\,(X)}{\partial x^2}=\langle D^2\psi(X)(e_x),e_x\rangle\leq\|D^2\psi\|_\infty,\qquad \frac{\partial^2\psi\,(X)}{\partial v^2}\leq\|D^2\psi\|_\infty,\tag{4.11}$$

and

$$k^{2s}\frac{\partial^2\psi\,(X)}{\partial z_k^2}\leq\|D^2\psi\|_\infty,\tag{4.12}$$

where for $D^2\psi(X):\mathcal{H}_{-s}\to L(\mathcal{H}_{-s},\mathcal{H}_{-s})$,

$$\|D^2\psi\|_\infty=\sup_{\mathcal{H}_{-s}}\|D^2\psi(Y)\|_{L(\mathcal{H}_{-s},\mathcal{H}_{-s})}.$$

In the RHS after the first equality of (4.9), the first four terms are bounded by, using (4.10), (4.11),

$$\begin{aligned}\int_{\mathcal{H}_{-s}}\Big|v\frac{\partial\psi(X)}{\partial x}\Big|+\frac{1}{m}\Big|\Phi'(x)\theta^R(x)\frac{\partial\psi(X)}{\partial v}\Big|&\\ +\frac{\gamma}{m}\Big|v\frac{\partial\psi(X)}{\partial v}\Big|+\frac{\gamma}{m^2}\Big|\frac{\partial^2\psi(X)}{\partial v^2}\Big|\,\mu(dX)&\end{aligned}\tag{4.13}$$

$$\leq\|D\psi\|_\infty\Big[\Big(1+\frac{\gamma}{m}\Big)\int_{\mathbb{R}}|v|\mu_v(dv)+\frac{1}{m}\int_{\mathbb{R}}|\Phi'(x)|\,\theta_R(x)e^{-\Phi(x)}dx\Big]+\frac{\gamma}{m^2}\|D^2\psi\|_\infty,$$

which is finite, by the definition of μ_v from (2.17) and the fact that θ_r as in (3.6) has compact support. For the first sum on the third line of (4.9), we estimate as follows.

$$\begin{aligned}&\sum_{k\geq 1}\int_{\mathcal{H}_{-s}}\left|\frac{\sqrt{c_k}}{m}z_k\frac{\partial\psi(X)}{\partial v}\right|\mu(dX)\\ &\qquad\leq\frac{\|D\psi\|_\infty}{m}\Big(\sum_{k\geq1}c_kk^{2s}\Big)^{1/2}\int_{\mathcal{H}_{-s}}\Big(\sum_{k\geq1}k^{-2s}z_k^2\Big)^{1/2}\mu(dX)\\ &\qquad\leq\frac{\|D\psi\|_\infty}{m}\Big(\sum_{k\geq1}c_kk^{2s}\Big)^{1/2}\int_{\mathcal{H}_{-s}}\|X\|_{-s}\mu(dX)<\infty,\end{aligned}\tag{4.14}$$

since by Assumption 3, $\sum_{k\geq1}c_kk^{2s}$ is finite and so is $\int_{\mathcal{H}_{-s}}\|X\|_{-s}\mu(dX)$, by the definition of μ, see (2.19). Similarly, for the second sum on the third line of (4.9), using (4.10) again, we infer

$$\begin{aligned}&\sum_{k\geq 1}\int_{\mathcal{H}_{-s}}\left|\sqrt{c_k}v\frac{\partial\psi(X)}{\partial z_k}\right|\mu(dX)\\ &\qquad\leq\int_{\mathcal{H}_{-s}}\Big(\sum_{k\geq1}\frac{c_k}{k^{2s}}\Big)^{1/2}\Big(\sum_{k\geq1}k^{2s}\left|\frac{\partial\psi(X)}{\partial z_k}\right|^2\Big)^{1/2}|v|\mu(dX)\\ &\qquad\leq\|D\psi\|_\infty\Big(\sum_{k\geq1}\frac{c_k}{k^{2s}}\Big)^{1/2}\int_{\mathcal{H}_{-s}}|v|\mu(dX)<\infty.\end{aligned}\tag{4.15}$$

For the first sum on the fourth line of (4.9), similar to (4.15), we invoke (4.10) again to see that

$$\sum_{k\geq 1}\int_{\mathcal{H}_{-s}}\left|\lambda_kz_k\frac{\partial\psi(X)}{\partial z_k}\right|\mu(dX)\leq\|D\psi\|_\infty\int_{\mathcal{H}_{-s}}\|X\|_{-s}\mu(dX)<\infty.\tag{4.16}$$

Lastly, we employ (4.12) to estimate the latter sum on the fourth line of (4.9),

$$\begin{aligned}\sum_{k\geq 1}\int_{\mathcal{H}_{-s}}\left|\lambda_k\frac{\partial^2\psi(X)}{\partial z_k^2}\right|\mu(dX)&=\int_{\mathcal{H}_{-s}}\left|\sum_{k\geq1}\lambda_kk^{-2s}k^{2s}\frac{\partial^2\psi(X)}{\partial z_k^2}\right|\mu(dX)\\ &\leq\|D^2\psi\|_\infty\int_{\mathcal{H}_{-s}}\sum_{k\geq1}\lambda_kk^{-2s}\mu(dX)<\infty.\end{aligned}\tag{4.17}$$

We can now apply Fubini Theorem on the Hilbert space $\mathcal{H}_{-s}$, see e.g. [28]. For $X\in\mathcal{H}_{-s}$, we write

$$X=P_{x,v}X+P_{x,v}^{\perp}X=xe_x+ve_v+Z,$$

where $P_{x,v}X = xe_x + ve_v$ is the projection on the subspace $\langle\{e_x, e_v\}\rangle$ and $P^\perp_{x,v}X = Z$ is the projection on $\langle\{e_x, e_v\}\rangle^\perp$. Then, μ can be decomposed as $\mu = \mu_{x,v} \times \mu^\perp_{x,v}$, where $\mu_{x,v} = \mu_x \times \mu_v$ is a measure on $P_{x,v}\mathcal{H}_{-s}$ and $\mu^\perp_{x,v} = \prod_{k\geq 1}\nu$ is a measure on $P^\perp_{x,v}\mathcal{H}_{-s}$. It follows that

$$\begin{aligned}
&\int_{\mathcal{H}_{-s}} I_{0,1}(X)\mu(dX)\\
&\qquad= \int_{\mathcal{H}_{-s}} v\frac{\partial\psi(X)}{\partial x} - \frac{1}{m}\Phi'(x)\theta^R(x)\frac{\partial\psi(X)}{\partial v}\mu(dX) \\
&\qquad= \int_{P^\perp_{x,v}\mathcal{H}_{-s}}\int_{\mathbb{R}^2} v\frac{\partial\psi(X)}{\partial x} - \frac{1}{m}\Phi'(x)\theta^R(x)\frac{\partial\psi(X)}{\partial v}\mu_{x,v}(dx,dv)\mu^\perp_{x,v}(dZ),
\end{aligned}\tag{4.18}$$

where we use Fubini's Theorem in the last implication. This is possible since we already established the absolute convergence in (4.13). Integrating by parts the first integral against $\mu_{x,v}$ yields

$$\begin{aligned}
\int_{\mathcal{H}_{-s}} I_{0,1}(X)\mu(dX) &= \int_{P^\perp_{x,v}\mathcal{H}_{-s}}\int_{\mathbb{R}^2} v\Phi'(x)\left(1-\theta^R(x)\right)\psi(X)\mu_{x,v}(dx,dv)\mu^\perp_{x,v}(dX)\\
&= \int_{\mathcal{H}_{-s}} v\Phi'(x)\left(1-\theta^R(x)\right)\psi(X)\mu(dX).
\end{aligned}\tag{4.19}$$

Similarly for $I_{0,2}$, we have

$$\begin{aligned}
&\int_{\mathcal{H}_{-s}} I_{0,2}(X)\mu(dX)\\
&\qquad= \int_{\mathcal{H}_{-s}}\left[-\frac{\gamma}{m}v\frac{\partial\psi(X)}{\partial v} + \frac{\gamma}{m^2}\frac{\partial^2\psi(X)}{\partial v^2}\right]\mu(dX)\\
&\qquad= \int_{P^\perp_{v}\mathcal{H}_{-s}}\int_{\mathbb{R}}\left[-\frac{\gamma}{m}v\frac{\partial\psi(X)}{\partial v} + \frac{\gamma}{m^2}\frac{\partial^2\psi(X)}{\partial v^2}\right]\mu_v(dv)\mu^\perp_v(d(xe_x+Z))\\
&\qquad= 0,
\end{aligned}\tag{4.20}$$

where we have employed (4.4) in the last implication. For the last two terms $I_{k,1}$, $I_{k,2}$, after integration by parts, we invoke (4.6), (4.5) respectively to obtain

$$\int_{\mathcal{H}_{-s}} I_{k,1}(X)\mu(dX) = 0, \qquad \int_{\mathcal{H}_{-s}} I_{k,2}(X)\mu(dX) = 0,\ k\geq 1. \tag{4.21}$$

Formula (4.7) now follows immediately from (4.9), (4.19), (4.20) and (4.21), thus completing the proof. □

We now show that μ is *almost invariant* under the cutoff system (3.7).

Proposition 17. *Let $R>0$. For every $\psi\in C_b^2(\mathcal{H}_{-s})$, there exists $\epsilon^R(\psi,t)>0$ such that*

$$\int_{\mathcal{H}_{-s}}\mathcal{P}^R(t)\psi(X)\mu(dX)=\int_{\mathcal{H}_{-s}}\psi(X)\mu(dX)+\varepsilon^R(\psi,t). \tag{4.22}$$

Furthermore, $\varepsilon^R(\psi,t)\to 0$ as $R\to\infty$.

Proof. Since equation (3.7) has a globally Lipschitz drift term, in view of Theorem 9.23 from [7], for every $\psi\in C_b^2(\mathcal{H}_{-s})$, $\mathcal{P}^R(t)\psi\in C_b^2(\mathcal{H}_{-s})$ satisfies the Kolmogorov backward equation, namely

$$\mathcal{P}^R(t)\psi(X)=\psi(X)+\int_0^t\mathcal{L}^R\mathcal{P}^R(r)\psi(X)dr. \tag{4.23}$$

Integrating both sides on $\mathcal{H}_{-s}$ with respect to μ gives

$$\int_{\mathcal{H}_{-s}}\mathcal{P}^R(t)\psi(X)\mu(dX)=\int_{\mathcal{H}_{-s}}\psi(X)\mu(dX)+\int_0^t\int_{\mathcal{H}_{-s}}\mathcal{L}^R\mathcal{P}^R(r)\psi(X)\mu(dX)dr. \tag{4.24}$$

We note that Fubini's theorem was applied to switch the order of integration in the double-integral term above. Indeed, from (4.13)-(4.16), we see that for all $r\in[0,t]$

$$\int_{\mathcal{H}_{-s}}\left|\mathcal{L}^R\mathcal{P}^R(r)\psi(X)\right|\mu(dX)\le c\left(\|D\mathcal{P}^R(r)\psi\|_\infty+\|D^2\mathcal{P}^R(r)\psi\|_\infty\right),$$

where $c>0$ is a constant independent of $R>0$. Furthermore, in view of Theorem 9.8 and Theorem 9.9 from [7], $\sup_{0\le r\le t}\|D\mathcal{P}^R(r)\psi\|_\infty$ and $\sup_{0\le r\le t}\|D^2\mathcal{P}^R(r)\psi\|_\infty$ are both finite. We thus infer that $\int_0^t\int_{\mathcal{H}_{-s}}\left|\mathcal{L}^R\mathcal{P}^R(r)\psi(X)\right|\mu(dX)dr<\infty$, which guarantees that the Fubini's Theorem is applicable. Now, since $\mathcal{P}(t)\psi\in C_b^2(\mathcal{H}_{-s})$ for all $t\ge 0$, Lemma 16 implies that

$$\begin{aligned}\int_{\mathcal{H}_{-s}}\mathcal{P}^R(t)\psi(X)\mu(dX)&=\int_{\mathcal{H}_{-s}}\psi(X)\mu(dX)\\&\quad+\int_0^t\int_{\mathcal{H}_{-s}}v\Phi'(x)\left(1-\theta^R(x)\right)\mathcal{P}^R(r)\psi(X)\mu(dX)dr.\end{aligned} \tag{4.25}$$

Let $\varepsilon^R(\psi,t)$ be given by

$$\varepsilon^R(\psi,t):=\int_0^t\int_{\mathcal{H}_{-s}}v\Phi'(x)\left(1-\theta^R(x)\right)\mathcal{P}^R(r)\psi(X)\mu(dX)dr. \tag{4.26}$$

It is clear that the integrand on the above RHS is dominated by $\|\psi\|_\infty |v\Phi'(x)|$ and that

$$\|\psi\|_\infty \int_0^t \int_{\mathcal{H}_{-s}} |v\Phi'(x)|\mu(dX)dr = t\|\psi\|_\infty \int_{\mathbb{R}} |v|\mu_v(dv) \int_{\mathbb{R}} |\Phi'(x)|\, \mu_x(dx) < \infty,$$

since μ_v is Gaussian and by Assumption 1, $\Phi'(x)e^{-\Phi(x)}$ is integrable. We additionally note that by the construction of local solutions, $X^R(r) \to X(r)$ as $R \to \infty$ a.s. It follows that $\mathcal{P}^R(r)\psi(X) \to \mathcal{P}(r)\psi(X)$, implying $v\Phi'(x)\left(1-\theta^R(x)\right)\mathcal{P}^R(r)\psi(X) \to 0$, since $\theta^R(x) \to 1$. We therefore apply the Dominated Convergence Theorem to infer that $\varepsilon^R(\psi,t) \to 0$, which completes the proof. □

With Proposition 17 in hand, we are ready to give the proof of Theorem 7.

Proof of Theorem 7. By taking $R \to \infty$ on both sides of (4.22), we obtain for all $\psi \in C_b^2(\mathcal{H}_{-s})$

$$\int_{\mathcal{H}_{-s}} \mathcal{P}(t)\psi(X)\mu(dX) = \int_{\mathcal{H}_{-s}} \psi(X)\mu(dX).$$

For $\varphi \in C_b(\mathcal{H}_{-s})$, approximating φ by a sequence $\{\psi_k\} \subset C_b^2(\mathcal{H}_{-s})$, we apply the Dominated Convergence Theorem to arrive at

$$\int_{\mathcal{H}_{-s}} \mathcal{P}(t)\varphi(X)\mu(dX) = \int_{\mathcal{H}_{-s}} \varphi(X)\mu(dX).$$

The proof is complete. □

5. Uniqueness of the invariant measure in the diffusive regime

In order to prove uniqueness of μ, we will construct an *asymptotic coupling*, following and applying the methods and ideas developed in [16]. Intuitively, this means that solutions started from different initial data have a positive probability of converging to one another as $t \to \infty$. Theorem 1.1 of [16] will then allow us to conclude there is only one ergodic invariant measure, thus uniqueness of μ follows by ergodic decomposition. For some more recent applications of this theory to SPDEs, we refer the reader to [10, 27].

For the reader's convenience, we briefly explain the framework of the *asymptotic coupling method* adapted to our setting, following [16, 10]. To begin, we denote by $\mathcal{H}_{-s}^{\mathbb{N}}$ the pathspace over $\mathcal{H}_{-s}$,

$$\mathcal{H}_{-s}^{\mathbb{N}} = \{\mathbf{U} : \mathbb{N} \to \mathcal{H}_{-s}\} = \{\mathbf{U} = (U_0, U_1, U_2, \dots) : U_i \in \mathcal{H}_{-s}\},$$

and let $\mathcal{P}(\mathcal{H}_{-s}^{\mathbb{N}}\times\mathcal{H}_{-s}^{\mathbb{N}})$ be the set of probability measures on $\mathcal{H}_{-s}^{\mathbb{N}}\times\mathcal{H}_{-s}^{\mathbb{N}}$. For any two measures M_1, M_2 on $\mathcal{H}_{-s}^{\mathbb{N}}$, we denote by $\widetilde{\mathcal{C}}(M_1,M_2)$ the collection of *asymptotically equivalent coupling* for M_1, M_2,

$$\widetilde{\mathcal{C}}(M_1,M_2)=\{\Gamma\in\mathcal{P}(\mathcal{H}_{-s}^{\mathbb{N}}\times\mathcal{H}_{-s}^{\mathbb{N}}):\Gamma\Pi_i^{-1}<<M_i, i=1,2\},\tag{5.1}$$

where $\Pi_1(u,v)=u$ and $\Pi_2(u,v)=v$. For any initial condition $X_0\in\mathcal{H}_{-s}$, let $\mathbf{X}=(X_0,X(1),X(2),\dots)$ be the corresponding solution path on $\mathcal{H}_{-s}^{\mathbb{N}}$ where $X(t)$ solves (3.1). Then the law of $\mathbf{X}$, denoted by $\delta_{X_0}\mathcal{P}^{\mathbb{N}}$, defines a probability measure on $\mathcal{H}_{-s}^{\mathbb{N}}$. Next, we introduce the set $\mathscr{D}$ given by

$$\mathscr{D}=\{(\mathbf{U},\mathbf{V})\in\mathcal{H}_{-s}^{\mathbb{N}}\times\mathcal{H}_{-s}^{\mathbb{N}}:\lim_{n\to\infty}\|U_n-V_n\|_{-s}=0\}.\tag{5.2}$$

Having introduced the above, we will seek to apply the following result (c.f. Corollary 2.2 of [16] and Corollary 2.1 [10]).

Theorem 18. *If for every pair X_0, $\widetilde{X}_0\in\mathcal{H}_{-s}$, there exists an element $\Gamma\in\widetilde{C}(\delta_{X_0}\mathcal{P}^{\mathbb{N}},\delta_{\widetilde{X}_0}\mathcal{P}^{\mathbb{N}})$ such that $\Gamma(\mathscr{D})>0$, then there exists at most one ergodic invariant measure for (3.1).*

The problem thus reduces to constructing such a coupling Γ. To this end, we introduce another process $\widetilde{X}(t)$ on $\mathcal{H}_{-s}$ satisfying the following shifted version of equation (3.1)

$$d\widetilde{X}(t)=L\widetilde{X}(t)\,dt+F(\widetilde{X}(t))\,dt+B\,dW(t)+B\,U(X(t),\widetilde{X}(t))\mathbf{1}\{t\le\tau\}\,dt.\tag{5.3}$$

In the above, $\widetilde{X}(0)=\widetilde{X}_0\in\mathcal{H}_{-s}$, τ is a stopping time and $U(X(t),\widetilde{X}(t))\in L^2(\Omega,\mathcal{W})$ is an adapted control depending on both $\widetilde{X}$ and the process X satisfying (3.1) with $X(0)=X_0\in\mathcal{H}_{-s}$. Here we recall that $\mathcal{W}$ is the auxiliary Hilbert space on which $W(t)$ evolves, [7]. Now notice that if we set

$$\widetilde{W}(t)=W(t)+\int_0^t U(X(r),\widetilde{X}(r))\mathbf{1}\{r\le\tau\}dr\tag{5.4}$$

and the control U and stopping time τ are such that, for some deterministic constant $C>0$,

$$\mathbb{P}\bigg\{\int_0^\infty\|U(X(t),\widetilde{X}(t))\mathbf{1}\{t\le\tau\}\|_{\mathcal{W}}^2\,dt<C\bigg\}=1,\tag{5.5}$$

then W and $\widetilde{W}$ are equivalent on $C([0,\infty);\mathcal{W})$. As a consequence, the processes X and $\widetilde{X}$ with $X(0)=\widetilde{X}(0)=\widetilde{X}_0\in\mathcal{H}_{-s}$ are mutually absolutely continuous on the infinitie time horizon $[0,\infty)$. As shown later in the proof of Theorem 10, our *coupling*

Γ is essentially the law of the pair $(X(\cdot, X_0), \widetilde{X}(\cdot, \widetilde{X}_0))$. However, in order for Γ to meet the requirement $\Gamma(\mathscr{D}) > 0$ from Theorem 18, by introducing the difference

$$\overline{X}(t) = X(t) - \widetilde{X}(t) = (\overline{x}(t), \overline{v}(t), \overline{z_1}(t), \dots), \tag{5.6}$$

we have to pick U and τ such that (5.5) is satisfied, $\mathbb{P}\{\tau = \infty\} > 0$ and

$$\|\overline{X}(t)\|_{-s} \to 0 \quad \text{as} \quad t \to \infty \quad \text{on the event} \quad \{\tau = \infty\}.$$

Thus, reemphasizing what was discussed above, we are constructing the control U and the stopping time τ such that we can drive two solutions of (3.1) with different initial data to one another as $t \to \infty$ on a set of positive probability.

To introduce our choice of U and τ, first observe that $\overline{X}$ satisfies $\overline{X}(0) = X_0 - \widetilde{X}_0$ and

$$d\overline{X}(t) = L\overline{X}(t)\, dt + \big[F(X(t)) - F(\widetilde{X}(t))\big]\, dt - B\, U(X(t), \widetilde{X}(t))\mathbf{1}\{t \le \tau\}\, dt. \tag{5.7}$$

Writing $X(t) = (x(t), v(t), z_1(t), \dots)$, $\widetilde{X}(t) = (\tilde{x}(t), \tilde{v}(t), \tilde{z}_1(t), \dots)$ and recalling the notation (5.6), we define for given $\lambda > 0$

$$\begin{aligned} u_0(X(t), \widetilde{X}(t)) = \frac{m}{\sqrt{2\gamma}} \bigg[&\Big(3\lambda - \frac{\gamma}{m}\Big)\overline{v}(t) + 2\lambda^2 \overline{x}(t) \\ &- \frac{1}{m}\left(\Phi'(x(t)) - \Phi'(\widetilde{x}(t))\right) - \frac{1}{m}\sum_{k\ge 1}\sqrt{c_k}\,\overline{z}_k(t)\bigg], \end{aligned} \tag{5.8}$$

and

$$U(X(t), \widetilde{X}(t)) = (0, u_0(X(t), \widetilde{X}(t)), 0, 0, \cdots). \tag{5.9}$$

Note that BU only possibly enacts control over the velocity difference $\bar{v}(t) = v(t) - \tilde{v}(t)$, and this is essentially done to gain control over nonlinear difference $\Phi'(x) - \Phi'(\tilde{x})$.

For a given $\kappa > 0$, we define the stopping time $\tau = \tau(\kappa)$ by

$$\tau = \inf_{t \ge 0}\left\{\int_0^t |u_0(X(s), \widetilde{X}(s))|^2\, ds \ge \kappa\right\}. \tag{5.10}$$

With these choices, note that on the event $\{t < \tau\}$, $\overline{X}(t) = (\bar{x}(t), \bar{v}(t), \bar{z}_1(t), \dots)$ satisfies the following system of equations

$$
\begin{aligned}
\frac{d\overline{x}(t)}{dt} &= \overline{v}(t), & \overline{x}(0) &= \overline{x}_0,\\
\frac{d\overline{v}(t)}{dt} &= -3\lambda\overline{v}(t) - 2\lambda^2\overline{x}(t), & \overline{v}(0) &= \overline{v}_0,\\
\frac{d\overline{z}_k(t)}{dt} &= -\lambda_k\overline{z}_k(t) + \sqrt{c_k}\overline{v}(t), & \overline{z}_k(0) &= (\overline{z}_k)_0.
\end{aligned}
\tag{5.11}
$$

Intuitively, the coefficient $\lambda > 0$ will be picked so that that $\|\overline{X}(t)\|_{\mathcal{H}_{-s}} \to 0$ as $t \to \infty$ on the event $\{\tau = \infty\}$. Hence the control induces the requisite dissipation, but we still need to see that we can pick $\kappa > 0$ so that (5.5) is satisfied and $\mathbb{P}\{\tau = \infty\} > 0$. Before turning to this issue, we make the following remark.

Remark 19. *(a) There is a significantly flexibility in the choice of u_0 in (5.8). One can of course choose other formulas for the coefficients of $\overline{x}(t)$ and $\overline{v}(t)$ in (5.8) as long as $\|\overline{X}(t)\|_{-s} \to 0$ as $t \to \infty$ on $\{\tau = \infty\}$.*

(b). The appearance of u_0 requires the drag constant γ be strictly positive. We note that for well-posedness (cf. Proposition 6) and the existence of invariant measures (cf. Theorem 7), γ can be zero.

With these observations, we state the following proposition which outlines the needed details to deduce unique ergodicity.

Proposition 20. *Under the Hypothesis of Theorem 10 and recalling $m, \gamma > 0$ from (2.7) and α, β as in (2.4), let $\lambda > 0$ be as in (5.8) and $\kappa > 0$ as in (5.10). Then there exist $\lambda = \lambda(\alpha, \beta) > 0$, $\kappa = \kappa(X_0, \widetilde{X}_0, \gamma, m, \alpha, \beta) > 0$ such that $\tau = \tau(\kappa)$ and U are such that*

(a) *Condition (5.5) is satisfied.*
(b) $\|\overline{X}(t)\|_{-s} \to 0$ *as* $t \to \infty$ *on* $\{\tau = \infty\}$.
(c) $\mathbb{P}\{\tau = \infty\} > 0$.

Before presenting the proof of Proposition 20, we now show how one can deduce unique ergodicity of (3.1) by combining Proposition 20 and Theorem 18, (see [16, 10] for further details).

Proof of Theorem 10. In view of Proposition 20 (a), the process $\widetilde{W}(t)$ defined in (5.4) is equivalent to the Wiener process $W(t)$ on $C([0, \infty); \mathcal{W})$ by Girsanov's theorem, (Theorem 10.4 from [7]). Moreover, the process $\widetilde{X}(\cdot, \widetilde{X}_0)$ solving (5.3) is absolutely continuous to the process $X(\cdot,\widetilde{X}_0)$ on $C([0,\infty);\mathcal{H}_{-s})$, (see e.g. [44]). It follows that the law Γ induced by

$$\big\{\big(X(nt,X_0),\widetilde{X}(nt,\widetilde{X}_0)\big): n=0,1,2,\dots\big\}$$

belongs to $\widetilde{\mathcal{C}}(\delta_{X_0}\mathcal{P}^{\mathbb{N}},\delta_{\widetilde{X}_0}\mathcal{P}^{\mathbb{N}})$. In addition, Proposition 20 (b) and (c) imply that $\Gamma(\mathscr{D})>0$ where D is given by (5.2). We therefore conclude unique ergodicity by virtue of Theorem 18, thus completing the proof. □

We now turn to the proof of Proposition 20. Parts (a) and (b) essentially follow by construction. Establishing part (c), however, requires a bit more work. To complete the proof, we need a crucial estimate on the potential Φ which relies on Lyapunov methods. To this end, for $N\in\mathbb{N}$ and $s\in\mathbb{R}$, we introduce $\Theta:\mathcal{H}_{-s}\to\mathbb{R}$ defined by

$$\Theta(X;s,N)=\frac{1}{m}\Phi(x)+\frac{1}{2}v^2+\frac{1}{2m}\sum_{k=1}^{N}z_k^2+\frac{1}{2}\sum_{k>N}k^{-2s}z_k^2. \tag{5.12}$$

In the diffusive regime, it turns out that $\Theta(X;s,N)$ can be chosen such that it satisfies a Lyapunov bound that is stronger than the bound on Ψ from Proposition 14. That is:

Proposition 21. *Let $\Theta(X;s,N)$ be defined as in (5.12). Then, under Assumptions 1 and Condition (D) of Assumption 3, there exists $N=N(m,\gamma,\alpha,\beta,s)\in\mathbb{N}$ sufficiently large such that, for some $a>0$, $\Theta(X):=\Theta(X;s,N)$ satisfies*

$$\sup_{X\in\mathcal{H}_{-s}}\mathcal{L}\Theta(X)\le a.$$

Proposition 21 will be established at the end of the section, but the proof follows a similar line of reasoning to that employed in the proof of Proposition 14.

Remark 22. *(a) In Assumption 3, the diffusive regime (D) requires $\alpha>1$ as opposed to $\alpha\in(0,1)$ and $\alpha=1$ in, respectively, the subdiffusive (SD) and critical (C) regimes. Recalling c_k, λ_k from (2.4), the condition $\alpha>1$ is needed so that the infinite sum*

$$\sum_{k\ge1}\frac{c_k}{2\lambda_k k^{-2s}}=\sum_{k\ge1}k^{-1-(\alpha-1)\beta+s}$$

converges, as shown later in (5.18) and (5.21). This convergence is critically employed in the proofs of Proposition 20 (c) and Propositon 21.

(b) The asymptotic behavior of λ_k as $k\to\infty$ presents a barrier to obtaining a stronger Lyapunov bound of the form

$$\mathcal{L}\Theta(X)\le -c\Theta(X)+a$$

in the proof of Proposition 21, where $c > 0$ is a constant. The above inequality, however, can be readily achieved in the finite-dimensional system (2.2), see [42]. With the appropriate support properties of the diffusion, such a bound implies geometric ergodicity. However, because we cannot see immediately why (22) holds in our infinite-dimensional system, suggests that perhaps the system relaxes to equilibrium slower than an exponential rate.

By combining the previous Proposition with the exponential martingale inequality, we obtain the following corollary.

Corollary 23. *Under Assumptions 1 and Condition* (D) *of Assumption 3, let $X(t) = (x(t), v(t), \dots)$ be the solution of (2.7) with initial condition $X_0 \in \mathcal{H}_{-s}$. Let Θ be the Lyapunov function defined in (5.12). Then there exists $\varepsilon = \varepsilon(m, \gamma, \alpha, \beta) > 0$ such that for every $\eta, r > 0$,*

$$\mathbb{P}\left\{\sup_{t\geq 0} \frac{e^{-\eta t}\Phi(x(t))}{m} - \Theta(X_0) - \frac{a}{\eta} \geq r\right\} \leq e^{-\varepsilon r}, \tag{5.13}$$

where a is as in the statement of Proposition 21.

The proof of Corollary 23 will also be given at the end of this section.

We now conclude Proposition 20 assuming the previous two results.

Proof of Proposition 20. We begin by showing part (a) of the result. In view of formulas (3.3) and (3.4), the norm of the control $U(t)$ in $\mathcal{W}$ satisfies

$$\|U(t)\|^2_{\mathcal{W}} = u_0(t)^2.$$

It thus follows by definition of τ that

$$\int_0^\infty \|U(X(t), \widetilde{X}(t))\mathbf{1}\{t \leq \tau\}\|^2_{\mathcal{W}}\, dt = \int_0^\infty |u_0(t)|^2 \mathbf{1}\{t \leq \tau\}\, dt = \kappa \qquad \mathbb{P}\text{−almost surely.}$$

Applying Theorem 10.4 from [7] finishes the proof of part (a).

To conclude part (b), for $t \leq \tau$ one can readily verify that the exact solution of (5.11) is given by

$$\begin{aligned}
\overline{x}(t) &= \Big(2\overline{x}_0 + \frac{\overline{v}_0}{\lambda}\Big)e^{-\lambda t} - \Big(\overline{x}_0 + \frac{\overline{v}_0}{\lambda}\Big)e^{-2\lambda t} \\
\overline{v}(t) &= -\left(2\lambda\overline{x}_0 + \overline{v}_0\right)e^{-\lambda t} + 2\left(\lambda\overline{x}_0 + \overline{v}_0\right)e^{-2\lambda t} \\
\overline{z}_k(t) &= e^{-\lambda_k t}\bigg[\left(\overline{z}_k\right)_0 + \sqrt{c_k}\int_0^t e^{\lambda_k r}\overline{v}(r)dr\bigg].
\end{aligned}$$

From this, it follows that

$$|\overline{x}(t)| \leq C_1 e^{-\lambda t}, \qquad |\overline{v}(t)| \leq C_2 e^{-\lambda t}, \tag{5.14}$$

where

$$C_1 := 3|\overline{x}_0| + \frac{3|\overline{v}_0|}{\lambda}, \qquad C_2 := 4\lambda|\overline{x}_0| + 4|\overline{v}_0|.$$

Combining these bounds, we thus obtain the following bound on $\overline{z}_k(t)$

$$|\overline{z}_k(t)| \leq e^{-\lambda_k t}\left[|(\overline{z}_k)_0| + C_2\sqrt{c_k}\int_0^t e^{(\lambda_k-\lambda)r}dr\right].$$

Choosing $\lambda = \lambda_1 + 1$ note that $\lambda - \lambda_k \geq 1$ for all $k \geq 1$ since $\lambda_k \downarrow 0$. With this choice of λ, it follows from the inequality above that for all $k \geq 1$,

$$|\overline{z}_k(t)| \leq e^{-\lambda_k t}\left(|(\overline{z}_k)_0| + C_2\sqrt{c_k}\right), \tag{5.15}$$

and hence by Young's inequality,

$$\overline{z}_k(t)^2 \leq 2e^{-2\lambda_k t}\left[(\overline{z}_k)_0^2 + C_2^2 c_k\right].$$

Thus putting it all together we find that

$$\begin{aligned}\|\overline{X}(t)\|_{-s}^2 &= \overline{x}(t)^2 + \overline{v}(t)^2 + \sum_{k\geq 1} k^{-2s}\overline{z}_k(t)^2 \\ &\leq \left(C_1^2 + C_2^2\right)e^{-\lambda t} + 2\sum_{k\geq 1} k^{-2s}\left(\overline{z}_k\right)_0^2 e^{-2\lambda_k t} + 2C_2^2\sum_{k\geq 1} k^{-2s}e^{-2\lambda_k t}c_k.\end{aligned}$$

Thus on the event $\{\tau = \infty\}$, it is now evident that $\|\overline{X}\|_{-s}^2 \to 0$ as $t \to \infty$ by applying the Monotone Convergence Theorem.

Turning to part (c) of the result, for any $R > 0$ consider the event E_R given by

$$E_R = \left\{\sup_{t\geq 0}\frac{e^{-\lambda t/2q}\Phi(x(t))}{m} - \Theta(X_0) - \frac{2qa}{\lambda} < R\right\}, \tag{5.16}$$

where q is the constant from Asumption 8. In view of Corollary 23 with $\eta = \lambda/2q$, E_R has positive probability provided $R = R(\gamma, m, \alpha, \beta) > 0$ is sufficiently large. We first claim that on E_R,

$$\int_0^\infty \|U(t)\|_{\mathcal{W}}^2\,dt \ \text{ is bounded almost surely.}$$

To see this, recall by definition of the control U that

$$\int_0^\infty \|U(t)\|_{\mathcal{W}}^2 dt = \int_0^\infty |u_0(t)|^2 dt.$$

Thus estimating $u_0(t)^2$, from (5.8) we have

$$\begin{aligned} u_0(t)^2 &\le \frac{2m^2}{\gamma}\bigg[\Big(3\lambda-\frac{\gamma}{m}\Big)^2\overline{v}(t)^2+4\lambda^4\overline{x}(t)^2 \\ &\qquad\qquad + \frac{1}{m^2}\Big(\sum_{k\ge1}\sqrt{c_k}|\overline{z}_k(t)|\Big)^2+\frac{1}{m^2}\left|\Phi'(x(t))-\Phi'(\widetilde{x}(t))\right|^2\bigg] \\ &= I_1(t)+I_2(t)+I_3(t)+I_4(t). \end{aligned}$$

For $I_1(t)+I_2(t)$, apply (5.14) to find

$$I_1(t)+I_2(t)\le\frac{2m^2}{\gamma}\left[\Big(3\lambda-\frac{\gamma}{m}\Big)^2C_2^2+4\lambda^4C_1^2\right]e^{-2\lambda t}=C_3e^{-2\lambda t}, \tag{5.17}$$

where $C_3:=\frac{2m^2}{\gamma}\left[\big(3\lambda-\frac{\gamma}{m}\big)^2C_2^2+4\lambda^4C_1^2\right]$. For $I_3(t)$, employ (5.15) to see that

$$\begin{aligned} I_3(t)&\le\frac{2}{\gamma}\Big(\sum_{k\ge1}\sqrt{c_k}e^{-\lambda_k t}\big(|(\overline{z}_k)_0|+C_2\sqrt{c_k}\big)\Big)^2 \\ &\le\frac{4}{\gamma}\Big(\sum_{k\ge1}\sqrt{c_k}e^{-\lambda_k t}|(\overline{z}_k)_0|\Big)^2+\frac{4C_2^2}{\gamma}\Big(\sum_{k\ge1}c_ke^{-\lambda_k t}\Big)^2 \\ &\le\frac{4\overline{X}_0^2}{\gamma}\sum_{k\ge1}\frac{c_ke^{-2\lambda_k t}}{k^{-2s}}+\frac{4C_2^2}{\gamma}K(t)^2, \end{aligned}$$

where the last inequality follows by Cauchy-Schwarz inequality since $K(t)=\sum_{k\ge1}c_ke^{-\lambda_k t}$ by definition. Lastly, to estimate $I_4(t)$, Assumption 8 and (5.14) together imply that

$$I_4(t)\le\frac{4}{\gamma}\overline{x}(t)^2\big(f\left(\overline{x}(t)\right)^2+\Phi(x(t))^{2q}\big)\le\frac{4C_1^2e^{-2\lambda t}}{\gamma}\Big(\sup_{|y|\le C_1}f\left(y\right)^2+\Phi(x(t))^{2q}\Big).$$

Now on the event E_R, we note that

$$\sup_{t\ge0}e^{-\lambda t}\Phi(x(t))^{2q}<\Big(m\Theta(X_0)+\frac{2qma}{\lambda}+mR\Big)^{2q}=:C_4.$$

Hence

$$\begin{aligned} I_4(t)&\le\frac{4C_1^2e^{-\lambda t}}{\gamma}\Big(e^{-\lambda t}\sup_{|y|\le C_1}f\left(y\right)^2+\sup_{t\ge0}e^{-\lambda t}\Phi(x(t))^{2q}\Big) \\ &\le\frac{4C_1^2}{\gamma}\Big(\sup_{|y|\le C_1}f\left(y\right)^2+C_4\Big)e^{-\lambda t} \\ &=C_5e^{-\lambda t}. \end{aligned}$$

Combining these bounds for I_1, I_2, I_3, I_4 shows that on E_R,

$$\int_0^\infty u_0(t)^2 dt \le \int_0^\infty I_1(t) + I_2(t) + I_3(t) + I_4(t)\, dt$$

$$\le \int_0^\infty C_3 e^{-2\lambda t} + \frac{4\overline{X}_0^2}{\gamma}\sum_{k\ge 1}\frac{c_k e^{-2\lambda_k t}}{k^{-2s}} + \frac{4C_2^2}{\gamma}K(t)^2 + C_5 e^{-\lambda t}\, dt \tag{5.18}$$

$$= \frac{4\overline{X}_0^2}{\gamma}\sum_{k\ge 1}\frac{c_k}{2\lambda_k k^{-2s}} + \int_0^\infty C_3 e^{-2\lambda t} + \frac{4C_2^2}{\gamma}K(t)^2 + C_5 e^{-\lambda t} dt.$$

We invoke Assumption 3 again to see that $\sum_{k\ge 1}\frac{c_k}{2\lambda_k k^{-2s}} < \infty$. Furthermore, in view of (2.6), $K(t)^2 \sim t^{-2\alpha}$ as $t\to\infty$, implying $K(t)^2$ is integrable. We thus infer from (5.18) a constant $C_6 = C_6(X_0, \widetilde{X}_0, \gamma, m, \alpha, \beta) > 0$ such that on E_R,

$$\int_0^\infty u_0(t)^2 dt \le C_6. \tag{5.19}$$

Finally, we choose $\kappa > C_6$ in the definition of $\tau = \tau(\kappa)$ forcing $E_R \subset \{\tau = \infty\}$. We therefore, conclude that $\mathbb{P}\{\tau_\kappa = \infty\} > 0$. The proof is thus complete. □

We now finish this section by giving the proofs of Proposition 21 and Corollary 23.

Proof of Proposition 21. We have

$$\mathcal{L}\Theta = -\frac{\gamma}{m}v^2 - \frac{1}{m}\sum_{k=1}^N \lambda_k z_k^2 - \sum_{k>N}\lambda_k k^{-2s} z_k^2 - \frac{1}{m}\sum_{k>N}\sqrt{c_k} z_k v$$

$$+ \sum_{k>N}\sqrt{c_k}k^{-2s} z_k v + \frac{\gamma}{m^2} + \frac{1}{m}\sum_{k=1}^N \lambda_k + \sum_{k>N}\lambda_k k^{-2s}.$$

Young's inequality combined with Cauchy-Schwarz inequality then gives

$$\frac{1}{m}\sum_{k>N}\sqrt{c_k} z_k v \le \frac{\gamma}{4m}v^2 + \frac{1}{\gamma m}\sum_{k>N}\frac{c_k}{k^{-2s}\lambda_k}\sum_{k>N}k^{-2s}\lambda_k z_k^2$$

and

$$\sum_{k>N}k^{-2s}\sqrt{c_k} z_k v \le \frac{\gamma}{4m}v^2 + \frac{m}{\gamma}\sum_{k>N}\frac{k^{-2s}c_k}{\lambda_k}\sum_{k>N}k^{-2s}\lambda_k z_k^2.$$

Combining the previous two inequalities with the first we obtain

$$\mathcal{L}\Theta \le -\frac{\gamma}{2m}v^2 - \frac{1}{m}\sum_{k=1}^N \lambda_k z_k^2 - a_1\sum_{k>N}k^{-2s}\lambda_k z_k^2 + a, \tag{5.20}$$

where

$$a := \frac{\gamma}{m^2} + \frac{1}{m}\sum_{k=1}^{N}\lambda_k + \sum_{k>N}\lambda_k k^{-2s}, \quad a_1 := 1 - \frac{1}{\gamma m}\sum_{k>N}\frac{c_k}{k^{-2s}\lambda_k} - \frac{m}{\gamma}\sum_{k>N}\frac{k^{-2s}c_k}{\lambda_k}. \tag{5.21}$$

We invoke Condition (D) of Assumption 3 again to see that

$$\sum_{k\geq 1}\lambda_k k^{-2s} = \sum_{k\geq 1}\frac{1}{k^{\beta+2s}} < \infty, \qquad \sum_{k\geq 1}\frac{c_k}{k^{-2s}\lambda_k} = \sum_{k\geq 1}\frac{1}{k^{1+(\alpha-1)\beta-2s}} < \infty,$$
$$\text{and} \qquad \sum_{k\geq 1}\frac{k^{-2s}c_k}{\lambda_k} = \sum_{k\geq 1}\frac{1}{k^{1+(\alpha-1)\beta+2s}} < \infty,$$

which implies that $a < \infty$ and that N can be chosen large enough such that $0 < a_1 < \infty$. We therefore conclude $\mathcal{L}\Theta \leq a$, which is the desired inequality. ☐

Proof of Corollary 23. . Fix $\eta > 0$ and apply Ito's Formula to $e^{-\eta t}\Theta(X(t))$ to find

$$d(e^{-\eta t}\Theta(X(t))) = -\eta e^{-\eta t}\Theta(X(t))\,dt + e^{-\eta t}\mathcal{L}\Theta(X(t))\,dt + dM_\eta(t) \tag{5.22}$$

where the martingale M_η satisfies

$$dM_\eta(t) = e^{-\eta t}\frac{\sqrt{2\gamma}}{m}v(t)\,dW_0(t) + \frac{e^{-\eta t}}{m}\sum_{k=1}^{N}\sqrt{2\lambda_k}z_k(t)\,dW_k(t)$$
$$+ e^{-\eta t}\sum_{k>N}\sqrt{2\lambda_k}k^{-2s}z_k(t)\,dW_k(t).$$

Note also that the quadratic variation process $\langle M_\eta\rangle$ has

$$d\langle M_\eta\rangle(t) = \frac{2\gamma e^{-2\eta t}}{m^2}v(t)^2\,dt + \frac{2e^{-2\eta t}}{m^2}\sum_{k=1}^{N}\lambda_k z_k(t)^2\,dt + 2e^{-2\eta t}\sum_{k>N}\lambda_k k^{-4s}z_k(t)^2\,dt. \tag{5.23}$$

We recall from (5.20) that

$$\mathcal{L}\Theta(X(t)) \leq -\frac{\gamma}{2m}v(t)^2 - \frac{1}{m}\sum_{k=1}^{N}\lambda_k z_k(t)^2 - a_1\sum_{k>N}k^{-2s}\lambda_k z_k(t)^2 + a, \tag{5.24}$$

where a, a_1 are defined in (5.21). Combining (5.22), (5.23) and (5.24), for every $\varepsilon > 0$ we obtain the estimate

$$\begin{aligned} d\left(e^{-\eta t}\Theta(X(t))\right) \leq ae^{-\eta t}dt + dM_\eta(t) - \frac{\varepsilon}{2}d\langle M_\eta\rangle(t) \\ - e^{-2\eta t}\Bigg[\frac{\gamma}{2m}v(t)^2 + \frac{1}{m}\sum_{k=1}^{N}\lambda_k z_k(t)^2 + a_1\sum_{k>N}k^{-2s}\lambda_k z_k(t)^2 \\ - \frac{\varepsilon}{2}\left(\frac{\gamma}{m^2}v(t)^2 + \frac{1}{m^2}\sum_{k=1}^{N}\lambda_k z_k(t)^2 + \sum_{k>N}\lambda_k k^{-4s} z_k(t)^2\right)\Bigg]dt. \end{aligned}$$

By choosing $\varepsilon = \varepsilon(\gamma, m, \alpha, \beta) > 0$ smaller if necessary, the bracket term on the above RHS is nonpositive. Hence

$$d(e^{-\eta t}\Theta(X(t))) \leq ae^{-\eta t}dt + dM_\eta(t) - \frac{\varepsilon}{2}d\langle M_\eta\rangle(t).$$

Integrating with respect to t we find

$$e^{-\eta t}\Theta(X(t)) - \Theta(X_0) \leq \int_0^\infty ae^{-\eta r}dr + M_\eta(t) - \frac{\varepsilon}{2}\langle M_\eta\rangle(t) = \frac{a}{\eta} + M_\eta(t) - \frac{\varepsilon}{2}\langle M_\eta\rangle(t).$$

Since $\Theta(X(t)) \geq \Phi(X(t))/m$ by the definition of $\Theta(X)$, we infer that

$$\frac{e^{-\eta t}\Phi(x(t))}{m} - \Theta(X_0) - \frac{a}{\eta} \leq M_\eta(t) - \frac{\varepsilon}{2}\langle M_\eta\rangle(t).$$

Invoking the exponential martingale inequality we obtain

$$\begin{aligned} \mathbb{P}\left\{\sup_{t\geq 0}\frac{e^{-\eta t}\Phi(x(t))}{m} - \Theta(X_0) - \frac{a}{\eta} \geq r\right\} &\leq \mathbb{P}\left\{\sup_{t\geq 0}\left[M_\eta(t) - \frac{\varepsilon}{2}\langle M_\eta\rangle(t)\right] \geq r\right\} \\ &\leq e^{-\varepsilon r}, \end{aligned}$$

thus completing the proof. □

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