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ANOMALOUS DIFFUSION AND THE GENERALIZED LANGEVIN EQUATION*

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Abstract. The generalized Langevin equation (GLE) is a stochastic integro-differential equation that is commonly used to describe the velocity of microparticles that move randomly in viscoelastic fluids. Such particles commonly exhibit what is known as anomalous subdiffusion, which is to say that their position mean-squared displacement (MSD) scales sublinearly with time. While it is common in the literature to observe that there is a relationship between the MSD and the memory structure of the GLE, and that there exist special cases where explicit solutions exist, this connection has never been fully characterized. Here, we establish a class of memory kernels for which the GLE is well-defined, we investigate the associated regularity properties of solutions, and we prove that large-time asymptotic behavior of the particle MSD is entirely determined by the tail behavior of the GLE's memory kernel.

Key words. viscoelastic diffusion, stationary processes, Fourier Abelian theorems

AMS subject classifications. 60G15, 60G20

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1. Introduction.

The generalized Langevin equation (GLE) is a stochastic integro-differential equation that is now commonly used to describe the velocity of microparticles diffusing in viscoelastic fluids. Introduced by Mori in 1965 [29] and Kubo in 1966 [21], and then popularized for modeling viscoelastic diffusion by Mason and Weitz in 1995 [26], the GLE is a balance-of-forces equation that features a prominent memory effect. Let $\{X(t)\}_{t\geq 0}$ and $\{V(t)\}_{t\in\mathbb{R}}$ be stochastic processes denoting a particle's time-dependent position and velocity. For the sake of simplicity, we will consider these processes to be one-dimensional, but this has no impact on our major findings. There are several perspectives on how the GLE can be derived from heat bath models [22, 20] or from principles of polymer physics [5] and viscoelastic fluid theory [8, 15]. With slight notational changes, we consider the version of the GLE that appears in [14], which has the most general form:

$$(1)\qquad m\mathrm{d}V(t) = -\lambda V(t) - \beta\int_{-\infty}^{t} K(t-s)V(s)\mathrm{d}s + \sqrt{\beta}F(t)\mathrm{d}t + \sqrt{2\lambda}\mathrm{d}W(t),$$

where m is the particle's mass, λ and β represent the particle's viscous and elastic drag coefficients, and $K:\mathbb{R}\to\mathbb{R}_+$ is a memory kernel that summarizes how the surrounding fluid stores kinetic energy from the particle and then acts back on the particle at a later time. The process $\{W(t)\}_{t\in\mathbb{R}}$ is a two-sided standard Brownian motion, while $\{F(t)\}_{t\in\mathbb{R}}$ is a mean zero, stationary, Gaussian process with covariance

$$(2)\qquad \mathbb{E}\left[F(t)F(s)\right] = K(t-s).$$

The fact that we require the covariance of $F(t)$ to be the same function as the memory kernel appearing in (1) is a manifestation of the fluctuation-dissipation relationship

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[21]. To have correct physical units, the coefficients of $F(t)$ and $\mathrm{d}W(t)$ should be $\sqrt{\beta k_B T}$ and $\sqrt{2\lambda k_B T}$, respectively, where k_B is Boltzmann's constant and T is the temperature of the system, but we will ignore this factor throughout this work. The reason there is a 2 in the coefficient of $\mathrm{d}W(t)$ but not in that of $F(t)$ is to satisfy equipartition of energy, as discussed in [12] and [14].

The GLE is one of a few qualitatively distinct mathematical models that can produce what is known as *anomalous diffusion*. A particle position process $X(t) := \int_0^t V(s)\mathrm{d}s$, $t \geq 0$ (sometimes referred to as the integrated GLE (iGLE)), is said to be *diffusive* if its mean-squared displacement (MSD), $\mathbb{E}\left[X^2(t)\right]$, satisfies $\mathbb{E}\left[X^2(t)\right] = Ct$ for some constant $C > 0$ for all time t. Any departure from being diffusive qualifies a process as exhibiting *anomalous diffusion*. Single particle tracking experiments for a wide variety of particles in biological fluids feature particles that exhibit anomalous *subdiffusion*, which is to say that for a large segment of time $\mathbb{E}\left[|X|^2(t)\right] \approx Ct^\alpha$ for some $\alpha \in (0,1)$ [6, 3, 11].

We will mostly concern ourselves with large-t behavior and whether an iGLE has the following property:

$$\text{(3)} \qquad \textit{Asymptotically subdiffusive } X(t): \ \ \mathbb{E}\left[X^2(t)\right] \sim t^\alpha \text{ as } t \to \infty,$$

where for two functions f and g, we say

$$f(t) \sim g(t) \text{ as } t \to \infty \text{ if for some } C \in (0,\infty), \lim_{t\to\infty} f(t)/g(t) = C.$$

The large-time MSD behavior of the iGLE is entirely determined by its memory kernel $K(t)$. To our knowledge, Morgado et al. (2002) [28] were the first to make this relationship explicit:

$$\text{(4)} \qquad \textit{Meta-theorem}\text{: for } \alpha \in (0,1), \ K(t) \sim t^{-\alpha} \implies \mathbb{E}\left[X^2(t)\right] \sim t^\alpha \text{ as } t \to \infty.$$

The argument presented by Morgado et al. was informal, and Kneller (2011) [18] later presented an attempt to make it rigorous. Both arguments rely on a chain of three relationships:

(i) relating the MSD to the autocovariance function (ACF, $r(t) := \mathbb{E}\left[V(t)V(s)\right]$);
(ii) relating the Laplace transform of the ACF to the Laplace transform of K;
(iii) relating the Laplace transform of K near zero to $K(t)$ itself for large t.

Relationship (i) follows from the classical formula [31]

$$\mathbb{E}\left[X^2(t)\right] = 2\int_0^t (t-s)r(s)\mathrm{d}s.$$

Relationship (iii) follows from the Hardy–Littlewood–Karamata (HLK) Tauberian theorem for Laplace transforms [4]. However, it has recently been shown that the proposed relationship (ii) is not valid [13]. The reason is that these arguments rely on a widely cited assumption that $\mathbb{E}\left[F(t)V(0)\right] = 0$ for all $t > 0$ in stationarity. This is, in fact, not the case for stationary solutions to the GLE. However, the assumption appears, for example, in the seminal works by Kubo (1966) [21], Mason (2000) [25], and Squires and Mason (2010) [37].

There are some special cases in which rigorous work has been done on the meta-theorem. In 2004, Kupferman [22] studied a version of the GLE where $\lambda = 0$ and the convolution integral in (1) is defined on the interval $[0,t]$ rather than $(-\infty, t]$. In this system, by assuming $K(t) = Ct^{-\alpha}$, the author derived an exact solution and demonstrated that the MSD scales like t^α. In 2008, Kou [20] presented the GLE ($\lambda = 0$) defined with the convolution over $(-\infty, t]$ and $K(t) = Ct^{-\alpha}$. Importantly, Kou shifted the analysis to a Fourier transform setting (more natural for studying a stationary process like V) and proved the meta-theorem holds in this special case. Later, in 2012, Didier et al. [2] introduced a condition on the Fourier transform of the spectral density of solutions (as the frequency tends to 0) that predicts the large-time scaling of the MSD. The limit theorem takes the form that there exist positive constants c and C such that $c \leq \lim_{t\to\infty} \mathbb{E}\left[X^2(t)\right]/t^\alpha \leq C$. However, the stated condition is not easy to interpret as a condition directly on $K(t)$.

In the work that follows we establish a large class of memory kernels $K(t)$ for which the GLE and iGLE are well-posed. We analyze regularity of the solutions and are able to characterize the large-t asymptotics of the MSD of $X(t)$ as follows: if $K(t)$ is integrable, then $X(t)$ is asymptotically diffusive; if $K(t)$ is not integrable, but has nice behavior for large t, then the meta-theorem (4) holds. In section 1.1 we lay out sufficient assumptions for $K(t)$ in two cases—in the first case (Assumption 1.1), when either $m > 0$ or $\lambda > 0$, and in the second case (Assumption 1.3), when $m = \lambda = 0$. Moreover, we describe some important memory kernel examples in the literature. In section 1.2 we provide a rigorous summary of our results including our version of the meta-theorem (4), namely Theorem 1.4.

1.1. The class of admissible memory functions $K(t)$.

The two primary examples of memory kernels from the literature are

$$\text{Sum of exponentials:} \quad K(t) = \sum_{k=1}^{n} c_k e^{-\lambda_k t} \quad [5,\ 8,\ 30,\ 23,\ 10],$$
$$\text{Power law:} \quad K(t) = c_\alpha t^{-\alpha} \ (\alpha \in (0,1)) \quad [22,\ 20]. \tag{5}$$

The coefficients of the sum of exponentials $\{c_k\}$ and the coefficient c_α are positive real numbers. We generalize these examples as follows.

ASSUMPTION 1.1. *Given $K : \mathbb{R} \to \mathbb{R}$, where $K(0)$ may be infinite, we assume that*

(I) a. *K is symmetric and positive for all nonzero t;*
 b. *$K(t) \to 0$ as $t \to \infty$ and is eventually decreasing;*
 c. *$K \in L^1_{loc}(\mathbb{R})$;*
 d. *The improper integral $\mathcal{K}_{\cos}(\omega) = \int_0^\infty K(t)\cos(\omega t)\,\mathrm{d}t$ is positive for all nonzero ω.*

Furthermore, either

(II) $K \in L^1$

or

(III) *$K \notin L^1$, but there exists $\alpha \in (0,1)$ such that $K(t) \sim t^{-\alpha}$ as $t \to \infty$.*

Remark 1.2. While (Ia), (Ib), and (Ic) are standard assumptions for covariance functions, (Id) may not be as familiar. In fact, condition (Id) is just a slightly stronger version of positive definiteness, which is a necessary condition for the covariance of any stationary stochastic process. Indeed, when K is positive definite either as a real-valued function (cf. Definition 2.1) or as a tempered distribution (cf. Definition 2.4), then $\mathcal{K}_{\cos}(\omega)$ has to be at least nonnegative for all nonzero ω; see Proposition 2.3 and Corollary 2.20. Condition (Id) makes the positivity of $\mathcal{K}_{\cos}$ strict.

Assumption 1.1 is sufficient as long as either $m > 0$ or $\lambda > 0$. If $m = \lambda = 0$, then we need to introduce stricter conditions. Most notably, K will need to be convex.

Assumption 1.3 (extension when $m = \lambda = 0$). *Given* $K : \mathbb{R} \to \mathbb{R}$, *where* $K(0)$ *may be infinite, we assume that*

(IV) $K \in C^2(0,\infty)$ *is convex and* $K''(t)$ *is monotone near the origin.*

Furthermore, either

(V) $K(0)$ *is finite and there exists* $\sigma_1 \in (0,1)$ *such that* $\lim_{t\to 0} t^{\sigma_1} K'(t) = 0$

or

(VI) $K(0)$ *is infinite, but there exists* $\sigma_2 \in (0,1)$ *such that* $\lim_{t\to 0} t^{\sigma_2} K(t) \in (0,\infty)$.

It has been noted in many places (recently in [27, 7]) that a sum of exponentials with sufficiently many terms can be used to approximate functions that have power-law behavior for large-t, but diverse behavior near the origin. As we note in section 2.6.1, the "closure" of the family of sum-of-exponential functions, namely the *completely monotone functions*, satisfies the conditions of Assumption 1.1.

1.2. Summary of results. In section 2, we lay out the mathematical foundation on which our main theorems are built. The results in sections 2.1–2.3 are a review of necessary definitions, notation, and results from classical stationary process theory (with a modest extension in section 2.4). Much of the work in section 3 is inspired from previous work by Soni and Soni (1975) [35]. Namely, we prove some Abelian theorems for improper Fourier transforms that are needed for our asymptotic analysis of the MSD in the subdiffusive case.

In section 4 we establish our notion of weak solutions for GLE/iGLE pairs, and in section 5 we provide conditions on $K(t)$ and the parameters m and λ that lead to continuous (or differentiable) versions of $V(t)$. The parameters m and λ play a prominent role here, and it does not matter whether the process is asymptotically diffusive or subdiffusive. We summarize these results as follows.

Suppose that $K(t)$ satisfies condition (I). Then if $m > 0$ or $\lambda > 0$, the GLE is well-posed and we find the following:

$$\begin{aligned} m > 0,\, \lambda > 0: &\quad V(t) \text{ is continuous a.s.}, \\ m > 0,\, \lambda = 0: &\quad V(t) \text{ is continuous a.s.}^{\dagger}, \\ m = 0,\, \lambda > 0: &\quad X(t) \text{ is continuous a.s.} \end{aligned} \tag{6}$$

In the last case, we understand the velocity process V in the sense of stationary random distributions. The † indicates that, in the ($m > 0$, $\lambda = 0$) case, stricter conditions can be placed on $K(t)$ so that $V(t)$ is, in fact, differentiable (see Theorem 5.6).

To address the ($m = \lambda = 0$) case, we must impose further conditions. Namely, suppose that, in addition to condition (I), $K(t)$ satisfies condition (IV) and either (V) or (VI). Then the GLE is well-posed and

$$m = 0,\, \lambda = 0: \quad X(t) \text{ is continuous a.s.} \tag{7}$$

Again, we understand V in the sense of stationary random distributions.

With these regularity results in hand, we proceed in section 6 to prove our *main theorem* on the dichotomy between being asymptotically diffusive or subdiffusive.

Theorem 1.4 (asymptotic behavior of the MSD). *Let* $\{V(t)\}_{t\in\mathbb{R}}$ *be a solution to the GLE in the sense defined in Definition* 4.1, *and let* $\{X(t)\}_{t\geq 0}$ *be the associated iGLE. If* $m > 0$ *or* $\lambda > 0$, *then*

$$\lim_{t\to\infty} \mathbb{E}\left[X^2(t)\right] \sim t^{\eta}, \text{ where } \eta = \begin{cases} 1 & \text{if } K(t) \text{ satisfies (I)} + \text{(II)}, \\ \alpha & \text{if } K(t) \text{ satisfies (I)} + \text{(III)}, \end{cases} \tag{8}$$

where, in the latter case, $\alpha \in (0,1)$ is the constant from condition (III)*.*

If $m = \lambda = 0$, then condition (I) *should be replaced with* (I) + *Assumption* 1.3*.*

This is our version of the meta-theorem (4), and the proof appears in section 6.

Finally, as has been noted in several places [27, 8, 30], a process might be asymptotically diffusive but nevertheless exhibit anomalous behavior over a very large time range. In section 7, we provide a rigorous definition for so-called *transient anomalous diffusion* and characterize one important setting in which it arises.

2. Mathematical preliminaries.

2.1. Tempered distributions and the Fourier transform.

For a function $f : \mathbb{R} \to \mathbb{C}$, we define the Fourier transform of f and its inverse as

$$\widehat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-it\omega} \mathrm{d}t \quad \text{and} \quad \check{f}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\omega) e^{it\omega} \mathrm{d}\omega.$$

We use $\mathcal{S}$ to denote the class of Schwartz functions and $\mathcal{S}'$ for the class of tempered distributions on $\mathcal{S}$. For $g \in \mathcal{S}'$, we write $\mathcal{F}[g]$ for the Fourier transform of g in $\mathcal{S}'$. That is, for all $\varphi \in \mathcal{S}$, it holds that

$$\langle g, \widehat{\varphi} \rangle = \langle \mathcal{F}[g], \varphi \rangle.$$

2.2. Positive definiteness.

We recall some basic definitions and theorems that can be found, for example, in the text by Cramér and Leadbetter [1].

DEFINITION 2.1. *A continuous function $r : \mathbb{R} \to \mathbb{C}$ is positive definite if the following holds:*

$$\sum_{j,k=1}^{n} r(t_j - t_k) z_j \overline{z}_k \geq 0$$

for any finite set of time points t_j and complex numbers z_j.

THEOREM 2.2 (Bochner's theorem). *A function $f(t)$ is positive definite if and only if it can be represented in the form*

$$f(t) = \int_{\mathbb{R}} e^{it\omega} \nu(\mathrm{d}\omega),$$

where ν is a positive finite Borel measure.

When the measure ν has a density $\widehat{f}$, i.e., the function f admits the formula $f(t) = \int_{\mathbb{R}} e^{it\omega} \widehat{f}(\omega) \mathrm{d}\omega$, then $\widehat{f}$ is called the spectral density. In fact, this is guaranteed by the first condition we impose on our memory kernels.

PROPOSITION 2.3. *Let f be a positive definite function satisfying* (Ib)*. Then, f admits the inverse Fourier formula*

$$f(t) = \frac{1}{\pi} \int_{\mathbb{R}} \widehat{f}(\omega) e^{it\omega} \mathrm{d}\omega, \tag{9}$$

where $\widehat{f}(\omega) = \int_0^\infty f(t) \cos(t\omega) \mathrm{d}t$.

The proof of Proposition 2.3 can be found in [16, Theorem 5.1]. The inversion formula (9) will be useful in section 5.1, where we investigate the differentiability of solutions to the GLE.

In order to make sense of the GLE in general, we will need the theory of stationary random distributions, introduced by Itô [17]. This requires an extension of the notion of positive definiteness to the tempered distributions.

DEFINITION 2.4. *A tempered distribution $f \in \mathcal{S}'$ is called positive definite if for any $\varphi \in \mathcal{S}$,*

$$\langle f, \varphi * \widetilde{\varphi} \rangle \geq 0,$$

where $\widetilde{\varphi}(x) = \varphi(-x)$.

Much as Bochner's theorem characterizes the positive definite functions, there is a characterization of positive definite tempered distributions as well.

THEOREM 2.5 ([17]). *A tempered distribution f is positive definite if and only if f admits a representation*

$$\langle f, \varphi \rangle = \int_{\mathbb{R}} \overline{\widehat{\varphi}(\omega)} \nu(\mathrm{d}\omega),$$

where ν is a nonnegative measure on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} \frac{\nu(\mathrm{d}x)}{(1+x^2)^k} < \infty \tag{10}$$

for some integer k.

Remark 2.6. Analogous to Theorem 2.2, when the measure ν in Theorem 2.5 is absolutely continuous to Lebesgue measure (i.e., if there exists a function $\widehat{f}$ such that $\nu(\mathrm{d}\omega) = \widehat{f}(\omega)\mathrm{d}\omega$), then $\widehat{f}$ is called the *spectral density* of the tempered distribution f.

2.3. Stationary random processes and stationary random distributions.

DEFINITION 2.7. *A stochastic process $\{F(t)\}_{t\in\mathbb{R}}$ is mean-square continuous and stationary if for all $t, s \in \mathbb{R}$,*
(a) $\mathbb{E}\left[|F(t)|^2\right] < \infty$ *and* $\lim_{h\to 0} \mathbb{E}\left[|F(t+h) - F(t)|^2\right] = 0$*;*
(b) $\mathbb{E}[F(t)] = a$ *for some constant a (we may assume $a = 0$); and*
(c) *the covariance function* $\mathbb{E}\left[F(t)\overline{F(s)}\right]$ *depends only on the difference* $(t-s)$*.*

This definition of stationarity is often called *stationary in the wide sense*, but we will simply call such processes *stationary.* The following connection between positive definite functions and covariance functions is explained, for example, in [1].

THEOREM 2.8. *A function $r(t)$ is positive definite if and only if it is the covariance function of some mean-square continuous stationary process $F(t)$, i.e.,*

$$r(t-s) = \mathbb{E}\left[F(t)\overline{F(s)}\right].$$

F can be chosen to be Gaussian.

The generalization of a stationary random *process* is a stationary random *distribution*, an idea introduced by Itô in 1954 [17]. Denote by τ_h the shift transform on $\mathcal{S}$, $\tau_h\varphi(x) := \varphi(x+h)$.

DEFINITION 2.9. *A linear functional $F : \mathcal{S} \to L^2(\Omega)$, the space of all random variables with finite variance, is called a stationary random distribution on $\mathcal{S}$ if for all $h \in \mathbb{R}$, $\varphi_1, \varphi_2 \in \mathcal{S}$,*

$$\mathbb{E}\left[\langle F, \tau_h\varphi_1\rangle \overline{\langle F, \tau_h\varphi_2\rangle}\right] = \mathbb{E}\left[\langle F, \varphi_1\rangle \overline{\langle F, \varphi_2\rangle}\right].$$

DEFINITION 2.10. *A process $\{\xi(t)\}_{t\in\mathbb{R}}$ is said to have orthogonal increments if for any $t_1 < t_2 \leq t_3 < t_4$, we have*

$$\mathbb{E}\left[\left(\xi(t_4) - \xi(t_3)\right)\overline{\left(\xi(t_2) - \xi(t_1)\right)}\right] = 0.$$

THEOREM 2.11 ([1]). *A process* $\{F(t)\}_{t\in\mathbb{R}}$ *is stationary if and only if there exists a stochastic process* $\{\xi(\omega)\}_{\omega\in\mathbb{R}}$ *with orthogonal increments such that for every* $t \in \mathbb{R}$,

$$F(t) = \int_{\mathbb{R}} e^{it\omega}\xi(\mathrm{d}\omega).$$

THEOREM 2.12 (characterization of stationary random distributions [17]). *A tempered distribution* r *is positive definite if and only if there exists a stationary random distribution* F *such that for all* $\varphi_1, \varphi_2 \in \mathcal{S}$,

$$\mathbb{E}\left[\langle F, \varphi_1\rangle\overline{\langle F, \varphi_2\rangle}\right] = \langle r, \varphi_1 * \widetilde{\varphi_2}\rangle.$$

The distribution r *is called the covariance distribution of* F.

Recalling Theorem 2.5, r can be represented by a nonnegative measure ν. We call ν the *spectral measure* of F.

Next, we recall the definition of *random measure*.

DEFINITION 2.13 ([17]). *Let* μ *be a nonnegative measure on* $\mathbb{R}$. *Denote by* $\mathcal{B}_\mu$ *the collection of all Borel sets* E *such that* $\mu(E) < \infty$. *A map* $\xi : \mathcal{B}_\mu \to L^2(\Omega)$ *is called a random measure with respect to* μ *if for* $E_1, E_2 \in \mathcal{B}_\mu$,

$$\mathbb{E}\left[\xi(E_1)\overline{\xi(E_2)}\right] = \mu(E_1 \cap E_2).$$

THEOREM 2.14. *Let* $\{F(\varphi)\}_{\varphi\in\mathcal{S}}$ *be a stationary random distribution with spectral measure* ν. *Then, there exists a random measure* ξ *that is defined with respect to* ν *such that*

$$\langle F, \varphi\rangle = \int_{\mathbb{R}} \overline{\widehat{\varphi}(\omega)}\xi(\mathrm{d}\omega).$$

2.4. An extension of the stationary random distributions. Let ν be a nonnegative measure on $\mathbb{R}$ satisfying (10) for some $k \in \mathbb{Z}$. Denote by $L^2(\nu)$ the Hilbert space of equivalence classes of nonrandom complex-valued functions g such that $\int_{\mathbb{R}} |g(s)|^2 \nu(\mathrm{d}s) < \infty$. Let ξ be a random measure with respect to ν as in Definition 2.13. For every $g \in L^2(\nu)$, the stochastic integral $\int_{\mathbb{R}} g(s)\xi(\mathrm{d}s)$ is a well-defined mean zero Gaussian random variable with

$$\mathbb{E}\left[\int_{\mathbb{R}} g_1(s)\xi(\mathrm{d}s)\overline{\int_{\mathbb{R}} g_2(s)\xi(\mathrm{d}s)}\right] = \int_{\mathbb{R}} g_1(s)\overline{g_2(s)}\nu(\mathrm{d}s).$$

See [17] for a detailed discussion.

As detailed above, there is a stationary random distribution $F : \mathcal{S} \to L^2(\Omega)$ whose spectral measure is ν. If we additionally have that ν is absolutely continuous with respect to Lebesgue measure, we may extend F to be an operator on $\mathcal{S}'$ as follows: for $g \in \mathcal{S}'$, let $\Phi : \mathcal{S}' \to L^2(\Omega)$ be defined as

$$\langle \Phi, g\rangle = \int_{\mathbb{R}} \overline{\mathcal{F}[g](\omega)}\xi(\mathrm{d}\omega). \tag{11}$$

The domain of Φ, denoted by $\mathrm{Dom}(\Phi)$, is the set of tempered distributions g such that its Fourier transform $\mathcal{F}[g]$ in $\mathcal{S}'$ is a function defined on $\mathbb{R}$ and that $\mathcal{F}[g] \in L^2(\nu)$. We stress that absolute continuity of ν with respect to Lebesgue measure is required in order to guarantee that the extension of F is well-defined. To be precise, we have the following lemma.

LEMMA 2.15. *Let $F : \mathcal{S} \to L^2(\Omega)$ be a stationary random distribution with spectral measure ν and associated random measure ξ. Let $\Phi : \mathcal{S}' \to L^2(\Omega)$ be the extension of F defined as in* (11)*. Assume further that ν is absolutely continuous to Lebesgue measure. Then, Φ is well-defined.*

Proof. Since ν is absolutely continuous with respect to Lebesgue measure, $\nu(\mathrm{d}\omega) = \widehat{r}(\omega)\mathrm{d}\omega$ for some function $\widehat{r}$. It suffices to show that the right-hand side (RHS) of (11) does not depend on the choice of $\mathcal{F}[g]$. To do this, suppose $\mathcal{F}_1[g]$ and $\mathcal{F}_2[g]$ are Fourier transforms of g in $\mathcal{S}'$; then it is known that they must agree almost everywhere. We then have a chain of implication

$$\begin{aligned}&\mathbb{E}\left|\int_{\mathbb{R}}\overline{\mathcal{F}_1[g](\omega)}\xi(\mathrm{d}\omega)-\int_{\mathbb{R}}\overline{\mathcal{F}_2[g](\omega)}\xi(\mathrm{d}\omega)\right|^2\\&\quad=\int_{\mathbb{R}}\left|\mathcal{F}_1[g](\omega)-\mathcal{F}_1[g](\omega)\right|^2\nu(\mathrm{d}\omega)=\int_{\mathbb{R}}\left|\mathcal{F}_1[g](\omega)-\mathcal{F}_1[g](\omega)\right|^2\widehat{r}(\mathrm{d}\omega)=0.\end{aligned}\tag{12}$$

It follows that two random variables $\int_{\mathbb{R}}\overline{\mathcal{F}_1[g](\omega)}\xi(\mathrm{d}\omega)$ and $\int_{\mathbb{R}}\overline{\mathcal{F}_2[g](\omega)}\xi(\mathrm{d}\omega)$ are equal a.s. We therefore conclude that Φ is well-defined. □

The function $\widehat{r}$ from (12) is called *the spectral density* of Φ.

DEFINITION 2.16 (the function-valued version of a stationary random distribution and its integral). *Let δ_t be the Dirac δ-distribution centered at t. If δ_t and $1_{[0,t]}$ are in* $\mathrm{Dom}(\Phi)$*, then we define*

$$V(t) := \langle \Phi, \delta_t\rangle \quad \textit{and} \quad X(t) := \langle \Phi, 1_{[0,t]}\rangle.\tag{13}$$

Note that $X(t)$ can be well-defined without $V(t)$.

The relationship between $V(t)$ and ν is characterized as follows.

LEMMA 2.17. *Let $\{\Phi(g)\}_{g\in\mathcal{S}'}$ be an extended stationary random distribution with spectral measure ν. Then the associated stationary random process $\{V(t)\}_{t\in\mathbb{R}}$ (as in Definition* 2.16*) is well-defined if and only if ν is a finite measure. In this situation, $X(t) = \int_0^t V(s)\mathrm{d}s$.*

Proof. The fact that the measure ν is finite is equivalent to

$$\mathcal{F}\left[\delta_t\right](\omega) = e^{-it\omega} \in L^2(\nu)$$

since $\int_{\mathbb{R}} \left|e^{-it\omega}\right|^2 \nu(\mathrm{d}\omega) = \int_{\mathbb{R}} \nu(\mathrm{d}\omega) < \infty$. This is precisely the condition for $\delta_t \in \mathrm{Dom}(V)$, which implies that $V(t)$ is well-defined.

Let ξ be the random measure with respect to ν in Definition 2.13. We note that the random measure ξ satisfies the orthogonal increments: for $t_1 < t_2 \le t_3 < t_4$,

$$\begin{aligned}\mathbb{E}\left[\left(\xi\left(t_4\right)-\xi\left(t_3\right)\right)\overline{\left(\xi\left(t_2\right)-\xi\left(t_1\right)\right)}\right]&=\nu\left(\{t_4\}\cap\{t_2\}\right)-\nu\left(\{t_4\}\cap\{t_1\}\right)\\&\quad+\nu\left(\{t_3\}\cap\{t_2\}\right)-\nu\left(\{t_3\}\cap\{t_1\}\right)=0,\end{aligned}$$

since ν is assumed to be absolute continuous with respect to Lebesgue measure. Consequently, $V(t)$ is actually a stationary Gaussian process. Indeed, thanks to the characterization Theorem 2.11, we have

$$V(t) = \langle V, \delta_t\rangle = \int_{\mathbb{R}} \overline{e^{-it\omega}}\xi(\mathrm{d}\omega) = \int_{\mathbb{R}} e^{it\omega}\xi(\mathrm{d}\omega).$$

Finally, the process $X(t)$ is given by

$$X(t) = \int_{\mathbb{R}} \overline{\mathcal{F}\left[1_{[0,t]}\right](\omega)}\xi(\mathrm{d}\omega) = \int_{\mathbb{R}} \overline{\int_0^t e^{-is\omega}\mathrm{d}s}\xi(\mathrm{d}\omega) \tag{14}$$

$$= \int_{\mathbb{R}}\int_0^t e^{is\omega}\mathrm{d}s\xi(\mathrm{d}\omega) = \int_0^t V(s)\mathrm{d}s.$$

The proof is thus complete. ∎

In general, one may understand $\langle V, g\rangle$ formally as the integral $\int_{\mathbb{R}} V(t)g(t)\mathrm{d}t$,

$$\langle V, g\rangle = \int_{\mathbb{R}} V(t)g(t)\mathrm{d}t = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{it\omega}\xi(\mathrm{d}\omega)g(t)\mathrm{d}t = \int_{\mathbb{R}} \overline{\widehat{g}(\omega)}\xi(\mathrm{d}\omega) = \int_{\mathbb{R}} \overline{\mathcal{F}\left[g\right](\omega)}\xi(\mathrm{d}\omega).$$

2.5. Sufficiency of conditions (I), (II), and (III). In this section, we establish that the conditions listed in Assumption 1.1 are sufficient for a function to be the covariance distribution of a stationary random distribution. In Lemma 2.18, we show that the improper Fourier sine and cosine transforms are well-defined for our class of memory kernels. Then, in Proposition 2.19, we show that our class of memory kernels are tempered distributions and express their Fourier transform in $\mathcal{S}'$ in terms of the improper Fourier cosine transform.

LEMMA 2.18. *Suppose that f satisfies conditions* (Ib) *and* (Ic) *of Assumption* 1.1. *Then, for $\omega \neq 0$, the improper integrals $\mathcal{F}_{\cos}(\omega) = \int_0^\infty f(t)\cos(t\omega)\mathrm{d}t$ and $\mathcal{F}_{\sin}(\omega) = \int_0^\infty f(t)\sin(t\omega)\mathrm{d}t$ are well-defined and continuous in ω, and*

$$\lim_{\omega\to\infty} \mathcal{F}_{\cos}(\omega) = \lim_{\omega\to\infty} \mathcal{F}_{\sin}(\omega) = 0. \tag{15}$$

Proof. The proof is essentially based on that of Lemma 1 in [35]. We rewrite it here because some of the estimates will be useful later. Fixing $A > 0$ large enough such that $f(t)$ decreases on $[A, \infty)$, we write

$$\int_0^\infty f(t)\cos(t\omega)\mathrm{d}t = \int_0^A f(t)\cos(t\omega)\mathrm{d}t + \int_A^\infty f(t)\cos(t\omega)\mathrm{d}t.$$

Because $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, the first integral on the RHS above is finite. Since $f > 0$ is decreasing on $t \geq A$, using the second mean value theorem, we have that for some $z \in (A, B)$

$$\int_A^B f(t)\cos(t\omega)\mathrm{d}t = f(A)\int_A^z \cos(t\omega)\mathrm{d}t + f(B)\int_z^B \cos(t\omega)\mathrm{d}t$$

$$\leq f(A)\left|\int_A^z \cos(t\omega)\mathrm{d}t\right| + f(B)\left|\int_z^B \cos(t\omega)\mathrm{d}t\right| \leq f(A)\frac{4}{\omega},$$

implying

$$\left|\int_A^\infty f(t)\cos(t\omega)\mathrm{d}t\right| \leq \frac{4f(A)}{\omega}. \tag{16}$$

Since $f(A) \downarrow 0$ as $A \to \infty$,

$$\lim_{A\to\infty}\left|\int_A^\infty f(t)\cos(t\omega)\mathrm{d}t\right| = 0.$$

It follows that $\int_0^\infty f(t)\cos(t\omega)\mathrm{d}t$ converges for all $\omega > 0$.

To demonstrate continuity, consider the limit as $\omega \to \omega_0 > 0$. Using inequality (16) gives

$$\left|\int_0^\infty f(t)\left[\cos(t\omega) - \cos(t\omega_0)\right]\mathrm{d}t\right| \leq \left|\int_0^A f(t)\left[\cos(t\omega) - \cos(t\omega_0)\right]\mathrm{d}t\right| + \frac{4f(A)}{\omega} + \frac{4f(A)}{\omega_0}.$$

Since $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, by the dominated convergence theorem, the integral on the RHS above converges to 0. Thus,

$$\limsup_{\omega\to\omega_0}\left|\int_0^\infty f(t)\left[\cos(t\omega) - \cos(t\omega_0)\right]\mathrm{d}t\right| \leq \frac{8f(A)}{\omega_0}.$$

Since A is arbitrarily large and $f \downarrow 0$, the continuity is evident. Likewise, $\mathcal{F}_{\sin}(\omega)$ is also well-defined and continuous for $\omega \neq 0$.

Finally, to demonstrate (15), observe that (16) implies

$$\left|\int_0^\infty f(t)\cos(t\omega)\mathrm{d}t\right| < \left|\int_0^A f(t)\cos(t\omega)\mathrm{d}t\right| + \frac{4f(A)}{\omega}. \tag{17}$$

By the Riemann–Lebesgue lemma, the first integral on the RHS above tends to 0 as $\omega \to \infty$. Since $f(A)$ is fixed, the second term also converges to 0, which demonstrates (15). ∎

Proposition 2.19. *Let f be a function satisfying condition* (I). *Suppose that either $f \in L^1(\mathbb{R})$ or $f \notin L^1(\mathbb{R})$ but there exists $\alpha \in (0,1)$ such that $t^\alpha f(t)$ is bounded near infinity. Then f is a tempered distribution and the following hold:*

(a) *The Fourier transform of f in $\mathcal{S}'$ is given by*

$$\mathcal{F}[f] = \widehat{f} = 2\mathcal{F}_{\cos}(\omega).$$

(b) *For any $\varphi \in \mathcal{S}$, the Fourier transform of $f^+ * \varphi$ in $\mathcal{S}'$ is given by*

$$\mathcal{F}\left[f^+ * \varphi\right](\omega) = \widehat{f^+}\widehat{\varphi} = \left(\mathcal{F}_{\cos}(\omega) - i\mathcal{F}_{\sin}(\omega)\right)\widehat{\varphi}(\omega),$$

where $f^+(t) = f(t)1_{[0,\infty)}(t)$.

Proof. The statement is straightforward when $f \in L^1(\mathbb{R})$. We are interested in the case when $t^\alpha f(t)$ is bounded near infinity. The proof is based on that of Theorem 1 in [35].

(a) Since f is locally integrable and decays to zero, it is clear that $f \in \mathcal{S}'$. We are left to show that for any $\phi \in \mathcal{S}$, there holds

$$\int_{\mathbb{R}} f(t)\widehat{\phi}(t)\mathrm{d}t = \int_{\mathbb{R}} 2\mathcal{F}_{\cos}(\omega)\phi(\omega)\mathrm{d}\omega. \tag{18}$$

For $k \in \mathbb{N}$, put $f_k(t) = f(t)1_{[0,k]}(|t|)$. We observe that $f_k \in L^1(\mathbb{R})$, which implies

$$\int_{\mathbb{R}} f_k(t)\widehat{\phi}(t)\mathrm{d}t = \int_{\mathbb{R}} \widehat{f_k}(\omega)\phi(\omega)\mathrm{d}\omega. \tag{19}$$

On one hand, as $k \to \infty$, $f_k(t)\widehat{\phi}(t)$ converges pointwise to $f(t)\widehat{\phi}(t)$ and is dominated by $|f\widehat{\phi}|$. We obtain, by the dominated convergence theorem,

$$\int_{\mathbb{R}} f_k(t)\widehat{\phi}(t)\mathrm{d}t \overset{k\to\infty}{\longrightarrow} \int_{\mathbb{R}} f(t)\widehat{\phi}(t)\mathrm{d}t. \tag{20}$$

On the other hand, it is clear from the proof of Lemma 2.18 that for all nonzero ω, $\widehat{f_k}(\omega)\phi(\omega)$ converges to $2\mathcal{F}_{\cos}(\omega)\phi(\omega)$. We are left to find a dominating function for $\widehat{f_k}$. To this end, there are two cases: $\omega > 1$ and $0 < \omega \le 1$. We fix A such that $f(t)$ is decreasing on $t \in [A.\infty)$.

Case 1. $\omega > 1$. We note that (16) still holds for $f(t)1_{[0,k]}(t)$ since $f(t)1_{[0,k]}(t)$ is decreasing on $t \in [A, \infty)$. We then estimate

$$\left|\widehat{f_k}(\omega)\right| = 2\left|\int_0^\infty f(t)1_{[0,k]}(t)\cos(\omega t)\mathrm{d}t\right| \le 2\int_0^A f(t)\mathrm{d}t + \frac{8f(A)}{\omega}. \tag{21}$$

Case 2. $0 < \omega \le 1$. We split the integral $\int_0^\infty f(t)1_{[0,k]}(t)\cos(t\omega)\mathrm{d}t$ into three parts:

$$\begin{aligned}\int_0^\infty f(t)1_{[0,k]}(t)\cos(\omega t)\mathrm{d}t &= \int_0^1 + \int_1^{A/\omega} + \int_{A/\omega}^\infty f(t)1_{[0,k]}(t)\cos(t\omega)\mathrm{d}t\\ &= I_0^k(\omega) + I_1^k(\omega) + I_2^k(\omega).\end{aligned}$$

For $I_0^k(\omega)$, we estimate

$$\left|I_0^k(\omega)\right| = \left|\int_0^1 f(t)1_{[0,k]}(t)\cos(\omega t)\mathrm{d}t\right| \le \int_0^1 f(t)\mathrm{d}t. \tag{22}$$

Next, by changing variable $z = t\omega$, we have

$$\begin{aligned}I_1^k(\omega) &= \frac{1}{\omega^{1-\alpha}}\int_\omega^A \left(\frac{z}{\omega}\right)^\alpha f\left(\frac{z}{\omega}\right)1_{[0,k]}\left(\frac{z}{\omega}\right)\frac{\cos(z)}{z^\alpha}\mathrm{d}z\\ &= \frac{1}{\omega^{1-\alpha}}\int_0^A 1_{[\omega,A]}(z)\left(\frac{z}{\omega}\right)^\alpha f\left(\frac{z}{\omega}\right)1_{[0,k]}\left(\frac{z}{\omega}\right)\frac{\cos(z)}{z^\alpha}\mathrm{d}z.\end{aligned}$$

Since $f(t)$ is continuous, $t^\alpha f(t)$ is bounded on $t \in [1, \infty)$. It follows that $I_1^k(\omega)$ is bounded by

$$I_1^k(\omega) = \frac{1}{\omega^{1-\alpha}}\int_0^A 1_{[\omega,A]}(z)\left(\frac{z}{\omega}\right)^\alpha f\left(\frac{z}{\omega}\right)1_{[0,k]}\left(\frac{z}{\omega}\right)\frac{\cos(z)}{z^\alpha}\mathrm{d}z \le \frac{c}{\omega^{1-\alpha}}\int_0^A \frac{1}{z^\alpha}\mathrm{d}z, \tag{23}$$

where $c > 0$ is a constant independent of k and ω. Finally, for $I_2^k(\omega)$, we invoke (16) to find

$$\begin{aligned}I_2^k(\omega) &= \int_{A/\omega}^\infty f(t)1_{[0,k]}(t)\cos(t\omega)\mathrm{d}t \le \frac{4}{\omega}f\left(\frac{A}{\omega}\right)1_{[0,k]}\left(\frac{A}{\omega}\right)\\ &= \frac{4A^\alpha}{\omega^{1-\alpha}}\left(\frac{A}{\omega}\right)^\alpha f\left(\frac{A}{\omega}\right)1_{[0,k]}\left(\frac{A}{\omega}\right) \le \frac{a_2}{\omega^{1-\alpha}},\end{aligned} \tag{24}$$

where in the last implication, we have employed again the fact that $t^\alpha f(t)$ is bounded on $[1, \infty)$. We now combine (21), (22), (23), and (24) to infer the existence of constants $c_1, c_2, c_3 > 0$ independent of k and $\omega \ne 0$ such that

$$\left|\widehat{f_k}(\omega)\right| \le \frac{c_1}{\omega^{1-\alpha}}1_{\{|\omega|\le 1\}}(\omega) + \frac{c_2}{\omega}1_{\{|\omega|>1\}}(\omega) + c_3.$$

Multiplying both sides of the above inequality by $|\phi(\omega)|$ yields

$$\left|\widehat{f}_k(\omega)\phi(\omega)\right| \leq \left(\frac{c_1}{\omega^{1-\alpha}}1_{\{|\omega|\leq 1\}}(\omega) + \frac{c_2}{\omega}1_{\{|\omega|>1\}}(\omega) + c_3\right)|\phi(\omega)|. \tag{25}$$

We observe now that the above RHS is integrable, which implies, by the dominated convergence theorem, that

$$\lim_{k\to\infty}\int_{\mathbb{R}}\widehat{f}_k(\omega)\phi(\omega)\mathrm{d}\omega = \int_{\mathbb{R}}2\mathcal{F}_{\cos}(\omega)(\omega)\phi(\omega)\mathrm{d}\omega. \tag{26}$$

We therefore infer (18) from (19), (20), and (26).

(b) Similarly to part (a), the Fourier transform of f^+ is given by

$$\mathcal{F}\left[f^+\right](\omega) = \widehat{f^+} = \mathcal{F}_{\cos}(\omega) - i\mathcal{F}_{\sin}(\omega).$$

Now, for $\psi\in\mathcal{S}$,

$$\langle f^+ * \phi, \widehat{\psi}\rangle = \int_{\mathbb{R}}\int_0^\infty f(s)\phi(t-s)\mathrm{d}s\widehat{\psi}(t)\mathrm{d}t.$$

In order to switch the order of integration, we have to check the Fubini condition. Using the fact that f eventually decreases, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\int_0^\infty f(s)\,|\phi(t-s)|\,\mathrm{d}s\left|\widehat{\psi}(t)\right|\mathrm{d}t\\
&\quad= \int_{\mathbb{R}}\left(\int_0^A f(s)\,|\phi(t-s)|\,\mathrm{d}s + \int_A^\infty f(s)\,|\phi(t-s)|\,\mathrm{d}s\right)\left|\widehat{\psi}(t)\right|\mathrm{d}t\\
&\quad\leq \int_{\mathbb{R}}\left(\|\phi\|_{L^\infty}\int_0^A f(s)\mathrm{d}s + f(A)\|\phi\|_{L^1}\right)\left|\widehat{\psi}(t)\right|\mathrm{d}t\\
&\quad= \|\widehat{\psi}\|_{L^1}\left(\|\phi\|_{L^\infty}\int_0^A f(s)\mathrm{d}s + f(A)\|\phi\|_{L^1}\right).
\end{aligned}$$

We thus obtain

$$\langle f^+ * \phi, \widehat{\psi}\rangle = \langle f^+, \widetilde{\phi} * \widehat{\psi}\rangle = \langle \widehat{f^+}, \widehat{\phi}\psi\rangle = \langle \widehat{f^+}\widehat{\phi}, \psi\rangle,$$

which completes the proof. ∎

Corollary 2.20. *Suppose $f:\mathbb{R}\to\mathbb{R}$ satisfies the hypotheses of Proposition* 2.19*; then f is the covariance distribution of a stationary random distribution whose spectral density is* $2\mathcal{F}_{\cos}(\omega)$.

Proof. It suffices to check that $f(t)$ is positive definite as a tempered distribution. For $\varphi\in\mathcal{S}$, we have

$$\langle f, \varphi * \widetilde{\varphi}\rangle = \int_{\mathbb{R}} f(t)\left(\varphi * \widetilde{\varphi}\right)(t)\mathrm{d}t = \int_{\mathbb{R}}2\mathcal{F}_{\cos}(\omega)\left|\varphi(\omega)\right|^2\mathrm{d}\omega \geq 0,$$

where the second and third implications follow from Proposition 2.19(a) and condition (Id), respectively. ∎

2.6. Examples of admissible memory kernels. Condition (I) requires that the Fourier cosine transform $\mathcal{F}_{\cos}$ be positive. A sufficient condition for K to guarantee positive Fourier cosine is that $K(t)$ be convex and locally integrable on $[0,\infty)$. To be precise, we record the following lemma, whose proof can be found in [38].

LEMMA 2.21. *Suppose that* $f \in L^1_{loc}([0,\infty))$, *convex on* $(0,\infty)$, *and decreasing to zero as* $t \to \infty$; *then for all* $\omega \neq 0$,

$$\int_0^\infty f(t)\cos(t\omega)\mathrm{d}t = \lim_{t\to\infty}\int_0^t f(s)\cos(s\omega)\mathrm{d}s > 0.$$

2.6.1. Sums of exponential functions. One family of memory functions that has proved useful in statistical analysis of viscoelastic diffusion is the generalized Rouse kernels [27, 23]. While these functions can have arbitrarily many terms, the family is fully described by three parameters, which makes the associated GLE amenable for parameter inference [23]. Let $p \geq 1$, $N \in \mathbb{N}$, and $\tau_0 > 0$ be given. Then we define the generalized Rouse kernels to be the set of functions of the form

$$K_N(t;p,\tau_0) = \frac{1}{N}\sum_{k=0}^{N-1} e^{-|\frac{t}{\tau_0}|(\frac{k}{N})^p}. \tag{27}$$

There is, in fact, an explicit form for the limit as N tends to infinity:

$$K_{\mathrm{Rouse}}(t;p,\tau_0) := \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1} e^{-|\frac{t}{\tau_0}|(\frac{k}{N})^p} = \int_0^1 e^{-|\frac{t}{\tau_0}|x^p}\mathrm{d}x. \tag{28}$$

The following proposition asserts that as N becomes larger, the tail of $K_N(t;p,\tau_0)$ behaves more and more like a power law of the form $t^{-1/p}$.

PROPOSITION 2.22. *Suppose that* $p \geq 1$, $N \in \mathbb{N}$, $\tau_0 > 0$. *Denote* $K_N = K_N(t;p,\tau_0)$ *and* $K = K_{\mathrm{Rouse}}(t;p,\tau_0)$, *where* $K_N(t;p,\tau_0)$ *and* $K_{\mathrm{Rouse}}(t;p,\tau_0)$ *are as in* (27) *and* (28), *respectively. Then,*

(a) $\lim_{N\to\infty}\sup_{t\in\mathbb{R}}|K_N(t) - K(t)| = 0$;

(b) $K(t) \sim t^{-1/p}$ *as* $t \to \infty$.

Proof. (a) Since K_n and K are even, it suffices to show that

$$\lim_{N\to\infty}\sup_{t\geq 0}|K_N(t) - K(t)| = 0. \tag{29}$$

We observe that $K_N(t)$, $K(t) \in [0,1]$ and that they are monotonically decreasing to zero on $t \in [0,\infty)$. The uniform convergence then follows from the pointwise convergence; see Exercise 13 in [32, p. 167].

(b) For $t > 0$, using a change of variable $y = \frac{t}{\tau_0}x^p$, $K(t)$ is equal to

$$K(t) = \frac{\tau_0^{1/p}}{pt^{1/p}}\int_0^{t/\tau_0} y^{1/p-1}e^{-y}\mathrm{d}y.$$

It follows immediately that

$$t^{1/p}K(t) = \frac{\tau_0^{1/p}}{p}\int_0^{t/\tau_0} y^{1/p-1}e^{-y}\mathrm{d}y \longrightarrow \frac{\tau_0^{1/p}}{p}\Gamma\left(\frac{1}{p}\right), \quad t\to\infty,$$

where $\Gamma(x)$ denotes the usual gamma function evaluated at x. ∎

The generalized Rouse kernel is a special case of a class of convex functions called the *completely monotone* functions.

DEFINITION 2.23. *A function $K : (0,\infty) \to \mathbb{R}$ is completely monotone if K is of class C^∞ and $(-1)^n K^{(n)}(t) \geq 0$ for all $n \geq 0$, $t > 0$. Denote the following:*
(a) *$\mathcal{CM}$ is the set of all completely monotone functions.*
(b) *$\mathcal{CM}_b$ is the set of all $K \in \mathcal{CM} \cap C[0,\infty)$.*

These functions are characterized by the following classical theorem.

THEOREM 2.24 (Hausdorff–Bernstein–Widder theorem [33]). *A function K is completely monotone if and only if K admits the formula*

$$K(t) = \int_0^\infty e^{-tx}\mu(\mathrm{d}x), \tag{30}$$

where μ is a positive measure on $[0,\infty)$.

2.6.2. Power law kernels. In [19] and [20], the authors considered the kernel

$$K_H(t) = 2H(2H-1)|t|^{2H-2}, \tag{31}$$

where $H \in (1/2, 1)$. Using the explicit Fourier transform of K_H, it is shown in [20] that the MSD satisfies $\mathbb{E}\left[X^2(t)\right] \sim t^{2-2H}$, which is subdiffusive.

To check that $K_H(t)$ verifies Assumption 1.1, we first note that $K_H(t)$ has power-law decay at infinity, which is consistent with condition (III). On the other hand, $K_H(t)$ is convex on $(0,\infty)$. Lemma 2.21 then implies that the improper Fourier cosine transform $\mathcal{K}_{\cos}(\omega) := \int_0^\infty K_H(t)\cos(t\omega)\mathrm{d}t$ is positive for every nonzero ω. It follows that $K_H(t)$ satisfies condition (I). We now can apply Corollary 2.20 to see that K_H is the covariance distribution of a stationary random distribution whose spectral density is $2\mathcal{K}_{\cos}(\omega)$.

Our theory of weak solutions in section 4 therefore applies to K_H. Furthermore, our result on the MSD in section 6 generalizes the result from [20], namely, the class of functions satisfying (I) + (III), of which K_H is a special case, leads to subdiffusive MSD.

2.6.3. An example of a class of nonconvex kernels. Given the examples we have presented so far, it might appear that convexity is required of K, but this is not the case. In Lemma 2.25 below, we show that our class of admissible memory kernels includes functions of the form $K(t) = \varphi\left(t^2\right)$, where $\varphi \in \mathcal{CM}_b$. Take $\beta > 0$; then $(1+t^2)^{-\beta/2}$ is a nonconvex yet admissible memory kernel because $(1+t)^{-\beta/2}$ is a completely monotone function. Note that when $\beta \in (0,1)$, the associated GLE is subdiffusive. In general, let μ be the representing measure of φ in Theorem 2.24; then $K(t) := \varphi(t^2)$ admits the representation

$$K(t) = \int_0^\infty e^{-t^2 x}\mu(\mathrm{d}x). \tag{32}$$

This function is not convex, but we are able to assert the following.

LEMMA 2.25. *Let $K(t) = \varphi\left(t^2\right)$, where $\varphi \in \mathcal{CM}_b$. Then, for every $\omega \neq 0$, $\mathcal{K}_{\cos}(\omega) > 0$.*

Proof. Replacing K with the formula (32), we have a chain of limits

$$\int_0^\infty K(t)\cos(t\omega)\mathrm{d}t = \lim_{A\to\infty}\int_0^A K(t)\cos(t\omega)\mathrm{d}t \tag{33}$$
$$= \lim_{A\to\infty}\int_0^A\int_0^\infty e^{-t^2x}\mu(\mathrm{d}x)\cos(t\omega)\mathrm{d}t = \lim_{A\to\infty}\int_0^\infty\int_0^A e^{-t^2x}\cos(t\omega)\mathrm{d}t\mu(\mathrm{d}x),$$

where in the last equality, we use the Fubini theorem to switch the order of integration. Now applying the second mean value theorem, we infer a $\xi\in(0,A)$ such that

$$\left|\int_0^A e^{-t^2x}\cos(t\omega)\mathrm{d}t\right| = \left|e^0\int_0^\xi\cos(t\omega)\mathrm{d}t\right| \le \frac{1}{\omega}.$$

We note that the representing measure μ is finite. Hence, by the dominated convergence theorem, we obtain

$$\lim_{A\to\infty}\int_0^\infty\int_0^A e^{-t^2x}\cos(t\omega)\mathrm{d}t\mu(\mathrm{d}x) = \int_0^\infty\int_0^\infty e^{-t^2x}\cos(t\omega)\mathrm{d}t\mu(\mathrm{d}x). \tag{34}$$

It follows from (33) and (34) that

$$\int_0^\infty K(t)\cos(t\omega)\mathrm{d}t = \int_0^\infty\left[\int_0^\infty e^{-t^2x}\cos(t\omega)\mathrm{d}t\right]\mu(\mathrm{d}x) = \int_0^\infty\frac{\sqrt{\pi}}{\sqrt{x}}e^{-\omega^2/4x}\mu(\mathrm{d}x),$$

which implies that $\int_0^\infty K(t)\cos(t\omega)\mathrm{d}t > 0$. ∎

In anticipation of the results that follow, we remark that since functions of this form satisfy Assumption 1.1 but not condition (IV) of Assumption 1.3, it is not clear whether the associated GLE is well-defined in the $(m=\lambda=0)$ case.

3. Abelian theorems for Fourier transforms.

In the subdiffusive case (with our specialized conditions), the behavior of the Fourier transform near the origin and near infinity can be characterized in a manner analogous to the Abelian theorems for the Laplace transform in the sense presented by Feller [4].

PROPOSITION 3.1. *Suppose that f satisfies the conditions* (Ib), (Ic), *and* (III) *from Assumption* 1.1. *Then*

$$\lim_{\omega\to0}\omega^{1-\alpha}\mathcal{F}_{\cos}(\omega)\in(0,\infty) \quad \textit{and} \quad \lim_{\omega\to0}\omega^{1-\alpha}\mathcal{F}_{\sin}(\omega)\in(0,\infty). \tag{35}$$

Remark 3.2. Proposition 3.1 is slightly different from Theorem 1 of [35], in which f is assumed to be finite at the origin. Our class of memory kernels need not satisfy this condition, recalling (31), for example. The technique that we use to treat the case where $K(t)$ is infinite at the origin is similar to the proof of Theorem 1.1 in [16].

Proof of Proposition 3.1. To establish (35), we shall improve the proof of Theorem 1 in [35]. Denote $c=\lim_{t\to\infty}t^\alpha f(t)$. By a change of variable, we have

$$\int_0^\infty f(t)\cos(\omega t)\mathrm{d}t = \int_0^\infty f\left(\frac{z}{\omega}\right)\frac{\cos(z)}{\omega}\mathrm{d}z.$$

Similar to the proof of Proposition 2.19(a), fixing A such that $f(t)$ is decreasing on $t\in[A,\infty)$, we split the above integral into three parts:

$$\omega^{1-\alpha}\int_0^\infty f(t)\cos(\omega t)\mathrm{d}t = \omega^{1-\alpha}\left[\int_0^\omega + \int_\omega^A + \int_A^\infty f\left(\frac{z}{\omega}\right)\frac{\cos(z)}{\omega}\mathrm{d}z\right]$$
$$= \omega^{1-\alpha}\left(I_0(\omega)+I_1(\omega)+I_2(\omega)\right).$$

For $I_0(\omega)$, by a change of variable again, we have

$$\omega^{1-\alpha}\,|I_0(\omega)| = \omega^{1-\alpha}\left|\int_0^1 f(t)\cos(\omega t)\mathrm{d}t\right| \le \omega^{1-\alpha}\int_0^1 f(t)\mathrm{d}t \overset{\omega\to 0}{\longrightarrow} 0, \tag{36}$$

since $f \in L^1_{loc}$. For $I_1(\omega)$, condition (III) combined with continuity implies that $t^\alpha f(t)$ is uniformly bounded on $t \in [1,\infty)$. It follows from the dominated convergence theorem that

$$\omega^{1-\alpha} I_1(\omega) = \int_0^A 1_{[\omega,A]}(z)\left(\frac{z}{\omega}\right)^\alpha f\left(\frac{z}{\omega}\right)\frac{\cos(z)}{z^\alpha}\mathrm{d}z \to c\int_0^A \frac{\cos(z)}{z^\alpha}\mathrm{d}z, \tag{37}$$

where $c = \lim_{t\to\infty} t^\alpha f(t)$. For the last term, $I_2(\omega)$, we invoke (16) again to find

$$\omega^{1-\alpha} I_2(\omega) \le \frac{4}{A^\alpha}\left(\frac{A}{\omega}\right)^\alpha f\left(\frac{A}{\omega}\right), \tag{38}$$

which implies

$$\limsup_{\omega\to 0} |I_2(\omega)| \le c\frac{4}{A^\alpha}. \tag{39}$$

Since A is chosen arbitrarily large, combining (36), (38), and (39), we obtain

$$\lim_{\omega\to 0}\omega^{1-\alpha}\mathcal{F}_{\cos}(\omega) = c\int_0^\infty \frac{\cos(z)}{z^\alpha}\mathrm{d}z. \tag{40}$$

We note that $\cos(z)/z^\alpha$ satisfies $\int_0^\infty \cos(z) z^{-\alpha}\mathrm{d}z \in (0,\infty)$; see [38]. We therefore obtain the Fourier cosine limit in (35). The Fourier sine limit is established using a similar argument. ∎

If we assume that f is nonincreasing, then Proposition 3.1 has an inverse statement as follows.

PROPOSITION 3.3 (Tauberian theorem for Fourier transform). *Suppose that f satisfies* (Ib) *and* (Ic) *from Assumption* 1.1 *and that f is nonincreasing. We further assume that $\mathcal{F}_{\cos}(\omega)/\omega \in L^1(1,\infty)$. Then,*

$$\lim_{\omega\to 0}\omega^{1-\alpha}\mathcal{F}_{\cos}(\omega) \in (0,\infty) \ \textit{ implies } \ \lim_{t\to\infty} t^\alpha f(t) \in (0,\infty).$$

We note that when f is assumed to be finite at the origin, the fraction $\mathcal{F}_{\cos}(\omega)/\omega$ automatically belongs to $L^1(1,\infty)$ since $\omega\mathcal{F}_{\cos}(\omega)$ is uniformly bounded in $0<\omega<\infty$ (cf. Lemma 1 in [35]). The result of Proposition 3.3 in this case was proved in Theorem 7 in [36]. Although Proposition 3.3 is not employed in later sections, we include it here for the completeness of the Abelian–Tauberian theorem. Also, the result still holds true if $\mathcal{F}_{\cos}$ is replaced by $\mathcal{F}_{\sin}$.

Proof of Proposition 3.3. For every $x > 0$, we consider the indicator function $h_x(t) = 1_{[0,x]}(t)$ whose Fourier cosine transform is given by

$$\int_0^\infty h_x(t)\cos(t\omega)\mathrm{d}t = \frac{\sin(x\omega)}{\omega}.$$

Since $f(t)$ and $h_x(t)$ are nonincreasing and $f(\cdot)h_x(\cdot) \in L^1(0,\infty)$, in view of Theorem 3.1 in [34], we have the following identity:

$$\int_0^x f(t)\mathrm{d}t = \int_0^\infty \mathcal{F}_{\cos}(\omega)\frac{\sin(x\omega)}{\omega}\mathrm{d}\omega.$$

The result now follows immediately from l'Hôpital's rule if we can show that

$$\frac{1}{x^{1-\alpha}}\int_0^\infty \mathcal{F}_{\cos}(\omega)\frac{\sin(x\omega)}{\omega}\mathrm{d}\omega \to c\in(0,\infty), \qquad x\to\infty.$$

To do this, we write the above integral as follows:

$$\begin{aligned}\frac{1}{x^{1-\alpha}}\int_0^\infty \mathcal{F}_{\cos}(\omega)\frac{\sin(x\omega)}{\omega}\mathrm{d}\omega &= \frac{1}{x^{1-\alpha}}\int_0^\infty \mathcal{F}_{\cos}\Big(\frac{z}{x}\Big)\frac{\sin(z)}{z}\mathrm{d}z\\ &= \frac{1}{x^{1-\alpha}}\int_0^x + \int_x^\infty \mathcal{F}_{\cos}\Big(\frac{z}{x}\Big)\frac{\sin(z)}{z}\mathrm{d}z\\ &= I_1(x)+I_2(x).\end{aligned}$$

By a change of variable again, we find

$$I_2(x) = \frac{1}{x^{1-\alpha}}\int_1^\infty \frac{\mathcal{F}_{\cos}(\omega)}{\omega}\sin(x\omega)\mathrm{d}\omega \to 0, \qquad x\to\infty,$$

following from the Riemann–Lebesgue lemma since $\mathcal{F}_{\cos}(\omega)/\omega \in L^1(1,\infty)$. With regard to I_1, we note that by the hypothesis on $\mathcal{F}_{\cos}$, the quantity $\big(\frac{z}{x}\big)^{1-\alpha}\mathcal{F}_{\cos}\big(\frac{z}{x}\big)$ is bounded on $z\in(0,x)$. It follows that

$$\begin{aligned}\frac{1}{x^{1-\alpha}}\mathcal{F}_{\cos}\Big(\frac{z}{x}\Big)\frac{\sin(z)}{z}1_{(0,x)}(z) &= \Big(\frac{z}{x}\Big)^{1-\alpha}\mathcal{F}_{\cos}\Big(\frac{z}{x}\Big)\frac{\sin(z)}{z^{2-\alpha}}1_{(0,x)}(z)\\ &\le c\Big[\frac{\sin(z)}{z^{1-\alpha}}1_{(0,1]}(z)+\frac{1}{z^{2-\alpha}}1_{(1,\infty)}(z)\Big],\end{aligned}$$

where $c>0$ is a constant that might change from line to line independent of x. The dominated convergence theorem then implies

$$I_1(x)\to c\int_0^\infty \frac{\sin(z)}{z^{2-\alpha}}\mathrm{d}z\in(0,\infty), \qquad x\to\infty.$$

Finally, collecting everything, we obtain the result. The proof is thus complete. ∎

For the large-ω asymptotic behavior of Fourier transform, we will need Assumption 1.3. This assumption is particularly useful in sections 4.4 and 6.2. We first observe that if a function f is convex and twice differentiable, the Fourier transform has the following representation, whose proof follows from integrating by parts.

LEMMA 3.4. *Suppose $f\in C^2(0,\infty)$ satisfies* (Ib) *and* (Ic). *Furthermore, we assume that f is convex and $\lim_{t\to 0^+} tf(t)=0$. Then for all $\omega>0$,*

$$\int_0^\infty f(t)\cos(t\omega)\mathrm{d}t = \frac{1}{\omega^2}\int_0^\infty f''(t)\left[1-\cos(t\omega)\right]\mathrm{d}t. \tag{41}$$

Proof. Since for all $t>0$, $f(t)$ is decreasing and $f''(t)\ge 0$, $f'(t)$ is increasing and negative. Now integration by parts gives

$$\begin{aligned}\int_0^\infty f(t)\cos(t\omega)\mathrm{d}t &= f(t)\frac{\sin(t\omega)}{\omega}\Big|_{t\to 0}^{t\to\infty} - \frac{1}{\omega}\int_0^\infty f'(t)\sin(t\omega)\mathrm{d}t\\ &= -\frac{1}{\omega}\int_0^\infty f'(t)\sin(t\omega)\mathrm{d}t,\end{aligned} \tag{42}$$

since $\lim_{t\to0^+} f(t)\sin(t\omega)/\omega = \lim_{t\to0^+} tf(t)\sin(t\omega)/(t\omega) = 0$ by assumption on f. For $t>0$, integration by parts once again gives

(43)
$$\begin{aligned}-\frac{1}{\omega}\int_t^\infty f'(x)\sin(x\omega)\mathrm{d}x &= f'(x)\frac{\cos(x\omega)}{\omega^2}\Big|_{x=t}^{x\to\infty} - \frac{1}{\omega^2}\int_t^\infty f''(x)\cos(x\omega)\mathrm{d}x\\ &= -f'(t)\frac{\cos(t\omega)}{\omega^2} - \frac{1}{\omega^2}\int_t^\infty f''(x)\cos(x\omega)\mathrm{d}x\\ &= -f'(t)\frac{\cos(t\omega)-1}{\omega^2} + \frac{1}{\omega^2}\int_t^\infty f''(x)\left[1-\cos(x\omega)\right]\mathrm{d}x.\end{aligned}$$

Sending $t\to0$, we have indeed

$$\lim_{t\to0} f'(t)\frac{\cos(t\omega)-1}{\omega^2} = \lim_{t\to0} t^2f'(t)\frac{\cos(t\omega)-1}{(t\omega)^2} = 0.$$

To show this, we use the fact that $f'<0$ and is decreasing on $t\in(0,\infty)$ to find

$$0 \ge t^2f'(t) = 2t\int_{t/2}^t f'(t)\mathrm{d}x \ge 2t\int_{t/2}^t f'(x)\mathrm{d}x = 2t\left(f(t)-f(t/2)\right) \xrightarrow{t\to0} 0. \tag{44}$$

It follows from (43) that

$$-\frac{1}{\omega}\int_0^\infty f'(x)\sin(x\omega)\mathrm{d}x = \frac{1}{\omega^2}\int_0^\infty f''(x)\left[1-\cos(x\omega)\right]\mathrm{d}x. \tag{45}$$

Finally, (41) follows from (42) and (45), which concludes the proof. ∎

We finally turn to asymptotic behavior of the Fourier transform. The following proposition is useful in section 4.4 when $m=\lambda=0$ in the GLE.

Proposition 3.5. *Suppose that $f(t)$ satisfies* (Ib) + (IV).

(a) *If $f(t)$ further satisfies* (V), *then*

$$\lim_{\omega\to\infty}\omega^{2-\sigma_1}\mathcal{F}_{\cos}(\omega) = 0 \quad \textit{and} \quad \lim_{\omega\to\infty}\omega\mathcal{F}_{\sin}(\omega) = f(0), \tag{46}$$

where σ_1 is the exponent from (V).

(b) *If $f(t)$ further satisfies* (VI), *then*

$$\lim_{\omega\to\infty}\omega^{1-\sigma_2}\mathcal{F}_{\cos}(\omega) \in (0,\infty) \quad \textit{and} \quad \lim_{\omega\to\infty}\omega^{1-\sigma_2}\mathcal{F}_{\sin}(\omega) \in (0,\infty), \tag{47}$$

where σ_2 is the exponent from (VI).

Proof. (a) Since $f(t)$ is convex on $t\in(0,\infty)$, it follows from Lemma 3.4 that

$$\mathcal{F}_{\cos}(\omega) = \frac{1}{\omega^2}\int_0^\infty f''(t)\left(1-\cos(t\omega)\right)\mathrm{d}t. \tag{48}$$

By a change of variable, $z=t\omega$, (48) is equivalent to

$$\omega^{2-\sigma_1}\mathcal{F}_{\cos}(\omega) = \int_0^\infty \left(\frac{z}{\omega}\right)^{1+\sigma_1} f''\left(\frac{z}{\omega}\right)\frac{1-\cos(z)}{z^{1+\sigma_1}}\mathrm{d}z. \tag{49}$$

We aim to use the dominated convergence theorem on the RHS above. Indeed, the integrand is dominated by $\frac{1-\cos(z)}{z^{1+\sigma_1}}$, which is integrable. To show this, we claim that

$t^{1+\sigma_1} f''(t)$ is uniformly bounded on $t \in (0,\infty)$. The only concerns are when t is near zero and when t is large. On one hand, notice that f'' is monotone near the origin by condition (IV). We write

$$-t^{\sigma_1} f'(t) = t^{\sigma_1} \int_t^{2t} f''(s)\mathrm{d}s - t^{\sigma_1} f'(2t) \geq t^{1+\sigma_1} f''(t) - t^{\sigma_1} f'(2t),$$

where we have assumed $f''(t)$ is increasing near the origin. By shrinking t to zero, we obtain $t^{1+\sigma_1} f''(t) \to 0$. A similar estimate also applies if we assume $f''(t)$ is decreasing, namely,

$$-t^{\sigma_1} f'(t) = t^{\sigma_1} \int_t^{2t} f''(s)\mathrm{d}s - t^{\sigma_1} f'(2t) \geq t^{1+\sigma_1} f''(2t) - t^{\sigma_1} f'(2t).$$

On the other hand, as $t \to \infty$, $f(t)/t^{1-\sigma_1} \to 0$. We employ the same trick to see that

$$-\frac{f(t)}{t^{1-\sigma_1}} = \frac{\int_1^t -f'(s)\mathrm{d}s}{t^{1-\sigma_1}} - \frac{f(1)}{t^{1-\sigma_1}} \geq -\frac{t-1}{t^{1-\sigma_1}} f'(t) - \frac{f(1)}{t^{1-\sigma_1}},$$

since $-f'(t)$ is increasing on the positive half-line. By taking $t \to \infty$, we obtain $t^{\sigma_1} f'(t) \to 0$. L'Hôpital's rule then implies

$$\lim_{t\to\infty} \frac{f''(t)}{-\sigma_1 t^{-1-\sigma_1}} = \lim_{t\to\infty} \frac{f'(t)}{t^{-\sigma_1}} = 0.$$

Now, from (49), sending ω to infinity, it follows from the dominated convergence theorem that

$$\lim_{\omega\to\infty} \omega^{2-\sigma_1} \mathcal{F}_{\cos}(\omega) = 0. \tag{50}$$

For the Fourier sine transform, we integrate by parts to find

$$\mathcal{F}_{\sin}(\omega) = \int_0^\infty f(t)\sin(t\omega)\mathrm{d}t = \frac{f(0)}{\omega} + \int_0^\infty f'(t)\frac{\cos(t\omega)}{\omega}\mathrm{d}t. \tag{51}$$

Multiplying through by ω, we obtain

$$\omega\mathcal{F}_{\sin}(\omega) = f(0) + \int_0^\infty f'(t)\cos(t\omega)\mathrm{d}t. \tag{52}$$

It suffices to show that $\lim_{\omega\to\infty}\int_0^\infty f'(t)\cos(t\omega)\mathrm{d}t = 0$. This in turn follows immediately from Lemma 2.18 since $f' \in L^1(0,\infty)$ and $-f'(t) \downarrow 0$ as $t \to \infty$.

(b) We use (49) again to write further

$$\begin{aligned}\omega^{1-\sigma_2}\mathcal{F}_{\cos}(\omega) &= \int_0^\infty \left(\frac{z}{\omega}\right)^{2+\sigma_2} f''\left(\frac{z}{\omega}\right)\frac{1-\cos(z)}{z^{2+\sigma_2}}\mathrm{d}z \\ &= \int_0^\omega + \int_\omega^\infty \left(\frac{z}{\omega}\right)^{2+\sigma_2} f''\left(\frac{z}{\omega}\right)\frac{1-\cos(z)}{z^{2+\sigma_2}}\mathrm{d}z \\ &= I_1(\omega) + I_2(\omega).\end{aligned} \tag{53}$$

We first claim that $\lim_{\omega\to\infty} I_2(\omega) = 0$. Indeed, by a change of variable again, i.e., $t = z/\omega$, we have

$$\begin{aligned}I_2(\omega) &= \frac{1}{\omega^{1+\sigma_2}}\int_1^\infty f''(t)\left(1-\cos(t\omega)\right)\mathrm{d}t \\ &\leq \frac{2}{\omega^{1+\sigma_2}}\int_1^\infty f''(t)\mathrm{d}t = \frac{-2f'(1)}{\omega^{1+\sigma_2}} \to 0.\end{aligned} \tag{54}$$

For $I_1(\omega)$, we write

$$I_1(\omega) = \int_0^\infty 1_{[0,\omega]}(z) \left(\frac{z}{\omega}\right)^{2+\sigma_2} f''\left(\frac{z}{\omega}\right) \frac{1-\cos(z)}{z^{2+\sigma_2}} \mathrm{d}z. \tag{55}$$

We wish to obtain from the dominated convergence theorem that

$$\lim_{\omega\to\infty} I_1(\omega) = \int_0^\infty \frac{1-\cos(z)}{z^{2+\sigma_2}} \mathrm{d}z \times \lim_{t\to 0} t^{2+\sigma_2} f''(t) \in (0,\infty). \tag{56}$$

To that end, we claim that $\lim_{t\to 0} t^{2+\sigma_2} f''(t) \in (0,\infty)$ and that $t^{2+\sigma_2} f''(t)$ is uniformly bounded on $t \in (0,1]$. The latter follows immediately from the former claim and the fact that $t^{2+\sigma_2} f''(t)$ is continuous. By condition (VI), $f(0^+) = \infty$. We apply l'Hôpital's Rule twice to see that

$$\lim_{t\to 0} \frac{f(t)}{t^{-\sigma_2}} = \lim_{t\to 0} \frac{-f'(t)}{t^{-1-\sigma_2}} = \lim_{t\to 0} \frac{f''(t)}{t^{-2-\sigma_2}} \in (0,\infty). \tag{57}$$

Thus, the integrand in (55) is dominated by

$$\sup_{t\in(0,1]} t^{2+\sigma_2} f''(t) \frac{1-\cos(z)}{z^{2+\sigma_2}},$$

as a function of z, which is integrable. We thus have shown (56). We finally combine (54) and (56) with (53) to obtain

$$\lim_{\omega\to\infty} \omega^{1-\sigma_2} \mathcal{F}_{\cos}(\omega) \in (0,\infty). \tag{58}$$

For the Fourier sine transform, we integrate by parts to obtain

$$\mathcal{F}_{\sin}(\omega) = \frac{1}{\omega} \int_0^\infty -f'(t)\left(1-\cos(t\omega)\right), \tag{59}$$

implying

$$\begin{aligned} \omega^{1-\sigma_2} \mathcal{F}_{\sin}(\omega) &= \int_0^\infty -\left(\frac{z}{\omega}\right)^{1+\sigma_2} f'\left(\frac{z}{\omega}\right) \left(\frac{1-\cos(z)}{z^{1+\sigma_2}}\right) \mathrm{d}z \\ &= \int_0^\omega + \int_\omega^\infty -\left(\frac{z}{\omega}\right)^{1+\sigma_2} f'\left(\frac{z}{\omega}\right) \left(\frac{1-\cos(z)}{z^{1+\sigma_2}}\right) \mathrm{d}z \\ &= I_3(\omega) + I_4(\omega). \end{aligned} \tag{60}$$

For $I_4(\omega)$, similarly to (54), we have the chain of implications

$$I_4(\omega) = \frac{1}{\omega^{\sigma_2}} \int_1^\infty -f'(t)\left(1-\cos(t\omega)\right) \mathrm{d}t \le \frac{1}{\omega^{\sigma_2}} \int_1^\infty -f'(t) \mathrm{d}t = \frac{f(1)}{\omega^{\sigma_2}} \to 0. \tag{61}$$

For $I_3(\omega)$, similarly to the argument we used to establish (56), we observe that $\lim_{t\to 0^+} t^{1+\sigma_2} f'(t) \in (0,\infty)$ thanks to (57) and that $t^{1+\sigma_2} f'(t)$ is bounded on $t \in (0,1]$ thanks to continuity. The dominated convergence theorem then implies

$$\lim_{\omega\to\infty} \omega^{1-\sigma_2} \mathcal{F}_{\sin}(\omega) \in (0,\infty). \tag{62}$$

The proof is complete. □

4. Weak solutions for the generalized Langevin equation. In order to define our notion of weak solutions for the GLE, we multiply (1) through by a test function $\varphi \in \mathcal{S}$ and integrate over the real line with respect to time. Formally, if we integrate by parts on the left-hand side and perform a change of variable in the convolution term, we arrive at the integral equation

$$-m \int_{\mathbb{R}} V(t)\varphi'(t)\mathrm{d}t = -\lambda \int_{\mathbb{R}} V(t)\varphi(t)\mathrm{d}t - \beta \int_{\mathbb{R}} V(t) \int_{\mathbb{R}} K^+(u)\varphi(t+u)\mathrm{d}u\mathrm{d}t$$
$$+ \sqrt{\beta} \int_{\mathbb{R}} F(t)\varphi(t)\mathrm{d}t + \sqrt{2\lambda} \int_{\mathbb{R}} \varphi(t)\mathrm{d}W(t),$$

where we have introduced the notation $K^+(t) := K(t)\,1_{\{t\geq 0\}}$. If we understand V, F, and the white noise process $\dot{W}$ as stationary random distributions in the sense of section 2, then we can write the GLE in its weak form

$$\langle V, -m\varphi' + \lambda\varphi + \beta\widetilde{K^+ * \widetilde{\varphi}}\rangle = \sqrt{2\lambda}\langle \dot{W}, \varphi\rangle + \sqrt{\beta}\langle F, \varphi\rangle, \tag{63}$$

where $\widetilde{f}(x) := f(-x)$. In this setting, the stationary random distributions $\dot{W}$ and F are defined in terms of their covariance structures:

$$\mathbb{E}\left[\langle \dot{W}, \varphi_1\rangle\langle \dot{W}, \varphi_2\rangle\right] = \int_{\mathbb{R}} \varphi_1(t)\varphi_2(t)\mathrm{d}t, \quad \mathbb{E}\left[\langle F, \varphi_1\rangle\overline{\langle F, \varphi_2\rangle}\right] = \int_{\mathbb{R}} K(t)\left(\varphi_1 * \widetilde{\varphi}_2\right)(t)\mathrm{d}t.$$

In other words, the spectral measure of $\dot{W}$ is Lebesgue measure, and the spectral measure of F is $\widehat{K}(\mathrm{d}\omega)$. In fact, we showed in section 2.5 that F has a spectral density, $2\mathcal{K}_{\cos}(\omega)$. See Corollary 2.20 in particular.

Definition 4.1. *Let ν be a nonnegative measure satisfying condition* (10)*, and let V be the operator associated with ν defined in* (11)*. Then V is a weak solution for* (63) *if V satisfies the following conditions:*
(a) *For all $\varphi \in \mathcal{S}$, $K^+ * \varphi$ belongs to* $\mathrm{Dom}(V)$.
(b) *For any $\varphi, \psi \in \mathcal{S}$, it holds that*

$$\mathbb{E}\left[\langle V, -m\varphi' + \lambda\varphi + \beta\widetilde{K^+ * \widetilde{\varphi}}\rangle\overline{\langle V, -m\psi' + \lambda\psi + \beta\widetilde{K^+ * \widetilde{\psi}}\rangle}\right]$$
$$= \mathbb{E}\left[\langle \sqrt{2\lambda}\dot{W} + \sqrt{\beta}F, \varphi\rangle\overline{\langle \sqrt{2\lambda}\dot{W} + \sqrt{\beta}F, \psi\rangle}\right].$$

The proof that weak solutions exist is sensitive to what is assumed about the parameters m and λ. We start with the most delicate proof, which is in the case $(m > 0, \lambda = 0)$. The cases $(m > 0, \lambda > 0)$ and $(m = 0, \lambda > 0)$ follow a similar argument (sections 4.2 and 4.3). The case $(m = 0, \lambda = 0)$ requires further assumptions about the memory kernel, and we handle this case in section 4.4.

4.1. Weak solutions when $m > 0$, but $\lambda = 0$. We begin by introducing the function $\widehat{r}$ in the following lemma.

Lemma 4.2. *Let K satisfy Assumption* 1.1*. Denote*

$$\widehat{r}(\omega) := \frac{\beta\widehat{K}(\omega)}{2\pi|mi\omega + \beta\widehat{K^+}(\omega)|^2}. \tag{64}$$

Then $\widehat{r}$ belongs to $L^1(\mathbb{R})$.

Proof. We can rewrite the formula for $\widehat{r}(\omega)$ as

$$\widehat{r}(\omega) = \frac{1}{2\pi} \times \frac{2\beta\mathcal{K}_{\cos}(\omega)}{[\beta\mathcal{K}_{\cos}(\omega)]^2 + [m\omega - \beta\mathcal{K}_{\sin}(\omega)]^2}. \tag{65}$$

Observing that $\widehat{r}$ is even, we need only consider $\omega \in [0,\infty)$. By Lemma 2.18, $\widehat{r}(\omega)$ is continuous on $(0,\infty)$. If $K \in L^1$, then

$$\lim_{\omega\to 0} \widehat{r}(\omega) = \frac{1}{\pi\beta\int_0^\infty K(t)\mathrm{d}t} < \infty.$$

If K is not in L^1 but satisfies (III), then Proposition 3.1 implies $\lim_{\omega\to0}\mathcal{K}_{\cos}(\omega) = \lim_{\omega\to0}\mathcal{K}_{\sin}(\omega) = \infty$. It follows from (65) that $\lim_{\omega\to0}\widehat{r}(\omega) = 0$. We see that in both cases, $\widehat{r}$ is locally integrable around zero. Now as ω tends to infinity, by Lemma 2.18, the numerator tends to 0, whereas the denominator is approximately $m^2\omega^2$, which implies that $\widehat{r}$ is integrable at infinity. We therefore conclude that $\widehat{r}$ belongs to $L^1(\mathbb{R})$. ∎

Lemma 4.2 implies that $\nu(\mathrm{d}\omega) = \widehat{r}(\omega)\mathrm{d}\omega$ satisfies (10) with $k = 0$. In view of Lemma 2.15, $\widehat{r}$ is the spectral density of some operator V defined as in (11). The following theorem asserts that V is indeed the weak solution of (63).

THEOREM 4.3. *Suppose that $m > 0$ and $\lambda = 0$ in* (63). *Let $K(t)$ satisfy Assumption* 1.1. *Then V is a weak solution for* (63) *if and only if the spectral measure ν satisfies $\nu(\mathrm{d}\omega) = \widehat{r}(\omega)\mathrm{d}\omega$, where $\widehat{r}$ is defined as in* (64).

Proof. $(\Rightarrow)$ Suppose V is a weak solution for (63). By Proposition 2.19(b),

$$\mathcal{F}\left[\widetilde{-m\varphi' + \beta K^+ * \widetilde{\varphi}}\right] = \mathcal{F}[-m\varphi'] + \overline{\mathcal{F}[\beta K^+ * \widetilde{\varphi}]} = \overline{im\omega}\widehat{\varphi} + \overline{\beta\widehat{K^+}\overline{\widehat{\varphi}}} = \overline{im\omega + \beta\widehat{K^+}}\widehat{\varphi}.$$

We thus have that

$$\begin{aligned}&\mathbb{E}\left[\langle V, \widetilde{-m\varphi' + \beta K^+ * \widetilde{\varphi}}\rangle\overline{\langle V, \widetilde{-m\psi' + \beta K^+ * \widetilde{\psi}}\rangle}\right]\\ &\quad= \int_{\mathbb{R}}\left(im\omega + \beta\widehat{K^+}(\omega)\right)\overline{\widehat{\varphi}}(\omega)\overline{im\omega + \beta\widehat{K^+}(\omega)}\widehat{\psi}(\omega)\nu(\mathrm{d}\omega)\\ &\quad= \int_{\mathbb{R}}\overline{\widehat{\varphi}}(\omega)\widehat{\psi}(\omega)\left|im\omega + \beta\widehat{K^+}(\omega)\right|^2\nu(\mathrm{d}\omega).\end{aligned}$$

One the other hand, by Proposition 2.19(a),

$$\int_{\mathbb{R}}\beta K(t)\left(\varphi * \widetilde{\psi}\right)(t)\mathrm{d}t = \int_{\mathbb{R}}\beta\widehat{K}(\omega)\frac{\overline{\widehat{\varphi}}(\omega)\widehat{\psi}(\omega)}{2\pi}\mathrm{d}\omega.$$

Since V is a weak solution, we obtain

$$\int_{\mathbb{R}}\overline{\widehat{\varphi}}(\omega)\widehat{\psi}(\omega)\left|im\omega + \beta\widehat{K^+}(\omega)\right|^2\nu(\mathrm{d}\omega) = \int_{\mathbb{R}}\beta\widehat{K}(\omega)\frac{\overline{\widehat{\varphi}}(\omega)\widehat{\psi}(\omega)}{2\pi}\mathrm{d}\omega.$$

Since all functions in $\mathcal{S}$ are the Fourier transform of some other Schwartz functions, we can rewrite the above formula as

$$\int_{\mathbb{R}}\varphi(\omega)\psi(\omega)\left|im\omega + \beta\widehat{K^+}(\omega)\right|^2\nu(\mathrm{d}\omega) = \int_{\mathbb{R}}\beta\widehat{K}(\omega)\frac{\varphi(\omega)\psi(\omega)}{2\pi}\mathrm{d}\omega.$$

Now we can choose $\{\varphi_k\}_{k\geq 1} \subset \mathcal{S}$, $\{\psi_k\}_{k\geq 1} \subset \mathcal{S}$ to be nonnegative and increasing up to $1_{[a,b]}$ and 1, respectively. The monotone convergence theorem then implies

$$\int_a^b \left| im\omega + \beta\widehat{K^+}(\omega)\right|^2 \nu(\mathrm{d}\omega) = \int_a^b \beta\frac{\widehat{K}(\omega)}{2\pi}\mathrm{d}\omega.$$

Since the equation above holds for any $-\infty < a < b < \infty$, we conclude that ν admits the Radon–Nikodym derivative $\nu(\mathrm{d}\omega) = \widehat{r}(\omega)\mathrm{d}\omega$.

($\Leftarrow$) Suppose $\nu(\mathrm{d}\omega) = \widehat{r}(\omega)$. To check the first condition of Definition 4.1, in view of Proposition 2.19(b), it suffices to show that

$$\int_{\mathbb{R}} \left|\mathcal{F}\left[K^+\right](\omega)\widehat{\phi}(\omega)\right|^2 \widehat{r}(\omega)\mathrm{d}\omega = \int_{\mathbb{R}} \left|\widehat{K^+}(\omega)\widehat{\phi}(\omega)\right|^2 \widehat{r}(\omega)\mathrm{d}\omega < \infty.$$

If $K \in L^1$, the inequality above is evident since $|\widehat{K^+}(\omega)\widehat{\phi}(\omega)|^2 \leq \|\widehat{K^+}\|^2_{L^\infty}\|\widehat{\phi}\|^2_{L^\infty}$ and $\widehat{r} \in L^1(\mathbb{R})$ by Lemma 4.2.

If K satisfies (III), as ω tends to infinity, Lemma 2.18 implies that $|\widehat{K^+}(\omega)\widehat{\phi}(\omega)|^2 \to 0$. It follows that $|\widehat{K^+}(\omega)\widehat{\phi}(\omega)|^2\widehat{r}(\omega)$ is dominated for sufficiently large ω by $\widehat{r}$, which is integrable. On the other hand, to control the integrand near zero, notice that

$$\begin{aligned} 2\pi\left|\widehat{K^+}(\omega)\widehat{\phi}(\omega)\right|^2\widehat{r}(\omega) &= \left|\widehat{\phi}(\omega)\right|^2 \frac{2\beta\mathcal{K}_{\cos}(\omega)\left(\left[\mathcal{K}_{\cos}(\omega)\right]^2 + \left[\mathcal{K}_{\sin}(\omega)\right]^2\right)}{\left[\beta\mathcal{K}_{\cos}(\omega)\right]^2 + \left[m\omega - \beta\mathcal{K}_{\sin}(\omega)\right]^2} \\ &= \frac{\left|\widehat{\phi}(\omega)\right|^2}{\omega^{1-\alpha}} \times \frac{2\beta\omega^{1-\alpha}\mathcal{K}_{\cos}(\omega)\left(\left[\omega^{1-\alpha}\mathcal{K}_{\cos}(\omega)\right]^2 + \left[\omega^{1-\alpha}\mathcal{K}_{\sin}(\omega)\right]^2\right)}{\left[\beta\omega^{1-\alpha}\mathcal{K}_{\cos}(\omega)\right]^2 + \left[m\omega^{2-\alpha} - \beta\omega^{1-\alpha}\mathcal{K}_{\sin}(\omega)\right]^2}. \end{aligned}$$

By Proposition 3.1, $\mathcal{K}_{\cos}(\omega)$ and $\mathcal{K}_{\sin}(\omega)$ can be controlled near the origin by $1/\omega^{1-\alpha}$. It follows that $|\widehat{K^+}(\omega)\widehat{\phi}(\omega)|^2\widehat{r}(\omega)$ is dominated by $\frac{|\widehat{\phi}(\omega)|^2}{\omega^{1-\alpha}}$ when ω is near zero. We conclude that $\widehat{K^+}\widehat{\phi}$ belongs to $L^2(\widehat{r})$. We thus have that

$$\begin{aligned} &\mathbb{E}\left[\langle V, -m\varphi' + \beta\widetilde{K^+ * \widetilde{\varphi}}\rangle\overline{\langle V, -m\psi' + \beta\widetilde{K^+ * \widetilde{\psi}}\rangle}\right] \\ &\quad= \int_{\mathbb{R}} \left(im\omega + \beta\widehat{K^+}(\omega)\right)\overline{\widehat{\varphi}}(\omega)\overline{im\omega + \beta\widehat{K^+}(\omega)}\widehat{\psi}(\omega)\widehat{r}(\omega)\mathrm{d}\omega \\ &\quad= \int_{\mathbb{R}} \beta\widehat{K}(\omega)\frac{\overline{\widehat{\varphi}}(\omega)\widehat{\psi}(\omega)}{2\pi}\mathrm{d}\omega \\ &\quad= \int_{\mathbb{R}} \beta K(t)\left(\varphi * \widetilde{\psi}\right)(t)\mathrm{d}t. \end{aligned}$$

The proof is thus complete. ∎

4.2. Weak solutions when $m > 0$ and $\lambda > 0$. Similarly to the preceding subsection, we introduce the following function $\widehat{r}$.

LEMMA 4.4. *Let $\widehat{r}$ be defined as*

$$\widehat{r}(\omega) = \frac{2\lambda + \beta\widehat{K}(\omega)}{2\pi|im\omega + \lambda + \beta\widehat{K^+}(\omega)|^2}. \tag{66}$$

Suppose K satisfies Assumption 1.1. Then $\widehat{r}$ belongs to $L^1(\mathbb{R})$.

Proof. The proof is the same as that of Lemma 4.2. ∎

Since $\widehat{r} \in L^1(\mathbb{R})$, it is the spectral density of some operator V defined as in (11). In the current situation where $\lambda > 0$, we will show that the weak solution V of (63) indeed admits $\widehat{r}$ defined as in (66) as the spectral density if we assume zero correlation between two stationary random distributions $\dot{W}$ and F.

Theorem 4.5. *Suppose that K satisfies Assumption* 1.1 *and that $m > 0$, $\lambda > 0$. Let V be a weak solution for* (63). *Then, the following statements are equivalent.*

(a) *The spectral measure ν admits the representation $\nu(\mathrm{d}\omega) = \widehat{r}(\omega)\mathrm{d}\omega$, where $\widehat{r}(\omega)$ is defined as in* (66).

(b) *For any $\varphi \in \mathcal{S}$, $\mathbb{E}\left[\langle W, \varphi\rangle\langle F, \varphi\rangle\right] = 0$.*

Proof. (a)⇒(b): On one hand, we have that

$$\mathbb{E}\left[\langle V, -m\varphi' + \lambda\varphi + \beta\widetilde{K^+ * \widetilde{\varphi}}\rangle\overline{\langle V, -m\psi' + \lambda\psi + \beta\widetilde{K^+ * \widetilde{\psi}}\rangle}\right]$$

$$\begin{aligned}&= \int_{\mathbb{R}} \left|im\omega + \lambda + \beta\widehat{K^+}(\omega)\right|^2 \widehat{r}(\omega)\overline{\widehat{\varphi}}\widehat{\psi}\mathrm{d}\omega \\ &= \int_{\mathbb{R}} \left(2\lambda + \beta\widehat{K}(\omega)\right)\frac{\overline{\widehat{\varphi}}\widehat{\psi}}{2\pi}\mathrm{d}\omega \\ &= 2\lambda\int_{\mathbb{R}} \varphi(t)\psi(t)\mathrm{d}t + \beta\int_{\mathbb{R}} K(t)\left(\varphi * \widetilde{\psi}\right)(t)\mathrm{d}t.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\mathbb{E}\left[\langle\sqrt{2\lambda}W + \sqrt{\beta}F, \varphi\rangle\overline{\langle\sqrt{2\lambda}W + \sqrt{\beta}F, \psi\rangle}\right] \\ &= 2\lambda\int_{\mathbb{R}} \varphi(t)\psi(t)\mathrm{d}t + \beta\int_{\mathbb{R}} K(t)\left(\varphi * \widetilde{\psi}\right)(t)\mathrm{d}t + \sqrt{2\lambda\beta}\,\mathbb{E}\left[\langle W, \varphi\rangle\overline{\langle F, \psi\rangle} + \overline{\langle W, \psi\rangle}\langle F, \varphi\rangle\right].\end{aligned}$$

Since V is a weak solution, we obtain $\mathbb{E}\left[\langle W, \varphi\rangle\overline{\langle F, \psi\rangle} + \overline{\langle W, \psi\rangle}\langle F, \varphi\rangle\right] = 0$, which is the same as $\mathbb{E}\left[\langle W, \varphi\rangle\langle F, \psi\rangle + \langle W, \psi\rangle\langle F, \varphi\rangle\right] = 0$, because they are real random variables. Replacing ψ with φ now implies (b).

(b)⇒(a): replacing φ with $\varphi + \psi$, (b) implies that

$$0 = \mathbb{E}\left[\langle W, \varphi + \psi\rangle\langle F, \varphi + \psi\rangle\right] = \mathbb{E}\left[\langle W, \varphi\rangle\langle F, \psi\rangle + \langle W, \psi\rangle\langle F, \varphi\rangle\right].$$

Reversing the order of the arguments above, we obtain

$$\int_{\mathbb{R}} \left|im\omega + \lambda + \beta\widehat{K^+}(\omega)\right|^2 \overline{\widehat{\varphi}}\widehat{\psi}\nu(\mathrm{d}\omega) = \int_{\mathbb{R}} \left(2\lambda + \beta\widehat{K}(\omega)\right)\frac{\overline{\widehat{\varphi}}\widehat{\psi}}{2\pi}\mathrm{d}\omega.$$

Using an approximating argument as in the proof of Theorem 4.3, we deduce that ν is absolutely continuous with respect to Lebesgue measure and that $\nu(\mathrm{d}\omega) = \widehat{r}(\omega)\mathrm{d}\omega$. The proof is complete. ∎

Remark 4.6. Since $\lambda > 0$, the Fourier cosine transform of K need not be strictly positive.

4.3. Weak solutions when $m = 0$ and $\lambda > 0$. In this case, the spectral density in formula (66) becomes

$$\widehat{r}(\omega) = \frac{2\lambda + \beta\widehat{K}(\omega)}{2\pi\left|\lambda + \beta\widehat{K^+}(\omega)\right|^2}. \tag{67}$$

LEMMA 4.7. *Let K satisfy Assumption* 1.1. *Then, $\widehat{r}$ defined as in* (67) *is the spectral density of a generalized operator V defined as in section* 2.4.

Proof. We note that $\widehat{r}$ is no longer integrable since $\lim_{\omega\to\infty}\widehat{r}(\omega)\in(0,\infty)$. However, using the assumption that $\mathcal{K}_{\cos}(\omega)\geq 0$, it follows from (67) that

$$\widehat{r}(\omega)=\frac{1}{2\pi}\times\frac{2\lambda+2\beta\mathcal{K}_{\cos}(\omega)}{\left(\lambda+\beta\mathcal{K}_{\cos}(\omega)\right)^2+\left(\beta\mathcal{K}_{\sin}(\omega)\right)^2}\leq\frac{1}{\pi\lambda}, \tag{68}$$

which implies that $\int_{\mathbb{R}}\frac{\widehat{r}(\omega)\mathrm{d}\omega}{1+\omega^2}<\infty$. In other words, the measure $\nu(\mathrm{d}\omega)=\widehat{r}(\omega)\mathrm{d}\omega$ satisfies (10) with $k=1$. In view of Lemma 2.15, $\widehat{r}$ is the spectral density of a generalized operator V defined as in (11). ∎

Similarly to Theorem 4.5, assuming zero correlation between $\dot{W}$ and F, we arrive at the following theorem.

THEOREM 4.8. *Suppose that K satisfies Assumption* 1.1 *and that $m=0$, $\lambda>0$. Let V be a weak solution for* (63). *Then, the following statements are equivalent:*
(a) *The spectral measure ν admits the representation $\nu(\mathrm{d}\omega)=\widehat{r}(\omega)\mathrm{d}\omega$, where $\widehat{r}$ is defined as in* (67).
(b) *For any $\varphi\in\mathcal{S}$, $\mathbb{E}\left[\langle W,\varphi\rangle\langle F,\varphi\rangle\right]=0$.*

Proof. The proof is the same as that of Theorem 4.5. ∎

4.4. Weak solutions when $m=0$ and $\lambda=0$. In this situation, the spectral density in formula (66) becomes

$$\widehat{r}(\omega)=\frac{\widehat{K}(\omega)}{\pi\beta\left|\widehat{K^+}(\omega)\right|^2}. \tag{69}$$

Because the structure of $\widehat{r}$ is quite different from the previous three cases, we need to impose Assumption 1.3 in addition to Assumption 1.1 on the memory kernel $K(t)$.

LEMMA 4.9. *Let $K(t)$ satisfy Assumptions* 1.1 *and* 1.3. *Then, $\widehat{r}$ defined as in* (69) *is the spectral density of a generalized operator V defined as in section* 2.4.

Proof. We need to check that $\nu(\mathrm{d}\omega)=\widehat{r}(\omega)\mathrm{d}\omega$ in this case satisfies (10). Indeed, we claim that inequality (10) holds with $k=1$, namely

$$\int_0^\infty\frac{\pi\beta}{2}\widehat{r}(\omega)\frac{1}{1+\omega^2}\mathrm{d}\omega=\int_0^\infty\frac{\mathcal{K}_{\cos}(\omega)}{\left(\mathcal{K}_{\cos}(\omega)^2+\mathcal{K}_{\sin}(\omega)^2\right)(1+\omega^2)}<\infty. \tag{70}$$

When ω is near zero, we have that

$$\frac{\pi\beta}{2}\widehat{r}(\omega)=\frac{\mathcal{K}_{\cos}(\omega)}{\mathcal{K}_{\cos}(\omega)^2+\mathcal{K}_{\sin}(\omega)^2}\leq\frac{1}{\mathcal{K}_{\cos}(\omega)}\to\frac{1}{\mathcal{K}_{\cos}(0^+)}, \tag{71}$$

which is either finite or zero depending on K being integrable or not, respectively. In other words, $\widehat{r}(\omega)$ is always bounded near the origin. The only concern now is when ω tends to infinity. Since K satisfies Assumption 1.3, Proposition 3.5 implies the existence of $\sigma\in(0,1)$ and $c(\sigma)>0$ such that for all ω sufficiently large,

$$\frac{\mathcal{K}_{\cos}(\omega)}{\mathcal{K}_{\cos}(\omega)^2+\mathcal{K}_{\sin}(\omega)^2}\leq c(\sigma)\omega^\sigma. \tag{72}$$

To show this, suppose K satisfies (V). Let σ_1 be the power constant from (V). We estimate

$$\frac{\mathcal{K}_{\cos}(\omega)}{\mathcal{K}_{\cos}(\omega)^2+\mathcal{K}_{\sin}(\omega)^2} \le \frac{\omega^{2-\sigma_1}\mathcal{K}_{\cos}(\omega)}{\left[\omega\mathcal{K}_{\sin}(\omega)\right]^2}\omega^{\sigma_1}. \tag{73}$$

We invoke (46) to find

$$\lim_{\omega\to\infty}\frac{\omega^{2-\sigma_1}\mathcal{K}_{\cos}(\omega)}{\left[\omega\mathcal{K}_{\sin}(\omega)\right]^2}=0 \tag{74}$$

and thus infer the constants σ and $c(\sigma)$ in (72), say $\sigma=\sigma_1$ and $c(\sigma)=1$. On the other hand, suppose K satisfies (VI). Let σ_2 be the power constant from (VI). Similarly to (73), we estimate

$$\frac{\mathcal{K}_{\cos}(\omega)}{\mathcal{K}_{\cos}(\omega)^2+\mathcal{K}_{\sin}(\omega)^2} \le \frac{\omega^{1-\sigma_2}\mathcal{K}_{\cos}(\omega)}{\left[\omega^{1-\sigma_2}\mathcal{K}_{\sin}(\omega)\right]^2}\omega^{1-\sigma_2}. \tag{75}$$

It follows from (47) that

$$\lim_{\omega\to\infty}\frac{\omega^{1-\sigma_2}\mathcal{K}_{\cos}(\omega)}{\left[\omega^{1-\sigma_2}\mathcal{K}_{\sin}(\omega)\right]^2}\in(0,\infty). \tag{76}$$

Setting $\sigma=1-\sigma_2$, $c=2\lim_{\omega\to\infty}\frac{\omega^{1-\sigma_2}\mathcal{K}_{\cos}(\omega)}{\left[\omega^{1-\sigma_2}\mathcal{K}_{\sin}(\omega)\right]^2}$, we obtain (72). We conclude that ν satisfies condition (10), which completes the proof. ∎

Using the same Definition 4.1 for weak solutions with $m=\lambda=0$, we have the following theorem.

THEOREM 4.10. *Suppose that $m=\lambda=0$. Let $K(t)$ satisfy Assumptions* 1.1 *and* 1.3*. Then V is a weak solution for* (63) *if and only if $\nu(\mathrm{d}\omega)=\widehat{r}(\omega)\mathrm{d}\omega$, where $\widehat{r}(\omega)$ is given by formula* (69).

Proof. The proof is essentially the same as that of Theorem 4.3. ∎

5. Regularity. We organize this section the same as section 4. We begin with the case ($m>0,\lambda=0$). The other two cases ($m>0,\lambda>0$) and ($m=0,\lambda>0$) are handled using similar arguments. The last case ($m=0,\lambda=0$) is treated differently. In addition, using classical theory of regularity of Gaussian processes, we will show that in the first case, with further assumptions on the memory kernel, $V(t)$ is differentiable a.s.

5.1. Regularity when $m>0$ and $\lambda=0$. We begin with the fact that the velocity $V(t)$ is well-defined as a stochastic process in time.

PROPOSITION 5.1. *Under the same hypotheses as in Theorem* 4.3*, let V be the weak solution of* (63)*. Then the process $V(t)=\langle V,\delta_t\rangle$ is well-defined.*

Proof. By Lemma 4.2, $\widehat{r}$ belongs to L^1. In view of Lemma 2.17, the spectral measure $\nu(\mathrm{d}\omega)=\widehat{r}(\omega)\mathrm{d}\omega$ is finite, which implies that $V(t)=\langle V,\delta_t\rangle$ is indeed a stationary, mean-square continuous Gaussian process. ∎

In order to establish the regularity of these Gaussian processes, we shall employ the following classic lemmas from Chapter 9.3 in [1].

LEMMA 5.2. *If a real stationary Gaussian process $\xi(t)$ with covariance function $k(t) = \int_{\mathbb{R}} e^{it\omega}\nu(\mathrm{d}\omega)$ satisfies*

$$\int_0^\infty [\log(1+\omega)]^a \, \nu(\mathrm{d}\omega) < \infty$$

for some $a > 3$, then $\xi(t)$ is equivalent to a process $\eta(t)$ which a.s. is continuous.

LEMMA 5.3. *If a real stationary Gaussian process $\xi(t)$ with covariance function $k(t) = \int_{\mathbb{R}} e^{it\omega}\nu(\omega)$ satisfies*

$$\int_0^\infty \omega^2 [\log(1+\omega)]^a \, \nu(\omega) < \infty$$

for some $a > 3$, then $\xi(t)$ is equivalent to a process $\eta(t)$ which is a.s. continuously differentiable.

We are now ready to assert the regularity of $V(t)$.

THEOREM 5.4. *Under the same hypotheses as in Theorem 4.3, let $V(t)$ be the Gaussian process defined in Proposition 5.1. Then $V(t)$ is continuous.*

Proof. The continuity of $V(t)$ will follow from Lemma 5.2 if it holds that

$$\int_0^\infty [\log(1+\omega)]^a \, \widehat{r}(\omega)\mathrm{d}\omega < \infty,$$

where $a > 3$ and $\widehat{r}$ is defined as in (65). The only issue here is when ω tends to infinity. However, for any $a > 3$, we note that

$$2\pi [\log(1+\omega)]^a \, \widehat{r}(\omega) = \frac{2\beta [\log(1+\omega)]^a \, \mathcal{K}_{\cos}(\omega)}{[\beta\mathcal{K}_{\cos}(\omega)]^2 + \omega^2 \left[m - \frac{\beta}{\omega}\mathcal{K}_{\sin}(\omega)\right]^2}.$$

In view of Lemma 2.18, $\lim_{\omega\to\infty} \mathcal{K}_{\cos}(\omega) = \lim_{\omega\to\infty} \mathcal{K}_{\sin}(\omega) = 0$. Hence, when ω is large, $[\log(1+\omega)]^a \, \widehat{r}(\omega)$ is dominated by $[\log(1+\omega)]^a / \omega^2$, which is integrable. We therefore conclude that $[\log(1+\omega)]^a \, \widehat{r}(\omega) \in L^1[0,\infty)$. ∎

As a consequence of $V(t)$ being continuous, we immediately obtain the following.

COROLLARY 5.5. *Under the same hypotheses as in Theorem 4.3, $X(t)$ is a.s. differentiable where $X(t) = \int_0^t V(s)\mathrm{d}s$.*

We finally assert a condition for differentiablity of $V(t)$.

THEOREM 5.6. *Under the same hypotheses as in Theorem 4.3, let $V(t)$ be as in Proposition 5.1. Assume further that K is positive definite and that for some $b > 3$*

$$K(0) - K(t) = O\left(|\log t|^{-b}\right), \quad t \to 0^+. \tag{77}$$

Then the Gaussian process $V(t)$ is a.s. continuously differentiable.

Proof. By Proposition 2.3, $\widehat{K}$ is integrable and K admits the inverse formula

$$K(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\omega}\widehat{K}(\omega)\mathrm{d}\omega = \int_0^\infty \cos(t\omega)\frac{\widehat{K}(\omega)}{\pi}\mathrm{d}\omega.$$

In view of Lemma 2 in [1, section 9.3], we deduce that for any $a < b$

$$\int_0^\infty |\log(1+\omega)|^a \widehat{K}(\omega)\mathrm{d}\omega < \infty. \tag{78}$$

By Proposition 2.19(a), the above inequality is equivalent to

$$\int_0^\infty |\log(1+\omega)|^a \mathcal{K}_{\cos}(\omega)\mathrm{d}\omega < \infty. \tag{79}$$

Now the differentiability of $V(t)$ follows immediately from Lemma 5.3 if we can show

$$\int_0^\infty \omega^2 \left[\log(1+\omega)\right]^a \widehat{r}(\omega)\mathrm{d}\omega < \infty,$$

which is the same as

$$\int_0^\infty \frac{\beta\omega^2 \left[\log(1+\omega)\right]^a \mathcal{K}_{\cos}(\omega)}{\left[\beta\mathcal{K}_{\cos}(\omega)\right]^2 + \omega^2\left[m - \frac{\beta}{\omega}\mathcal{K}_{\sin}(\omega)\right]^2}\mathrm{d}\omega < \infty. \tag{80}$$

On one hand, when ω is near the origin, the integrand in (80) is dominated by $\widehat{r}$, which is integrable by virtue of Lemma 4.2. On the other hand, when ω becomes large, reasoning as in the proof of Theorem 4.3, we see that, the integrand is dominated by $\left[\log(1+\omega)\right]^a \mathcal{K}_{\cos}(\omega)$, which is also integrable thanks to (79). We therefore obtain (80), which in turns implies the differentiability of $V(t)$. The proof is complete. ∎

5.2. Regularity when $m > 0$ and $\lambda > 0$.

PROPOSITION 5.7. *Under the same hypotheses as in Theorem* 4.5*, let V be the weak solution of* (63)*. Then, the velocity process $V(t) = \langle V, \delta_t\rangle$ is well-defined.*

Proof. In view of Lemma 2.17, we need to check that the spectral measure $\nu(\mathrm{d}\omega) = \widehat{r}(\omega)\mathrm{d}\omega$ is finite, where $\widehat{r}$ is defined in (66). This in turns follows immediately from Lemma 4.4. ∎

We assert that $V(t)$ is always continuous in this case.

THEOREM 5.8. *Under the same hypotheses as in Theorem* 4.5*, let $V(t)$ be the Gaussian process from Proposition* 5.7*. Then $V(t)$ is continuous.*

Proof. The proof is similar to that of Theorem 5.4. ∎

We immediately obtain the differentiability of the particle position process $X(t)$.

COROLLARY 5.9. *Under the same hypotheses as in Theorem* 4.5*, $X(t)$ is a.s. differentiable where $X(t) = \int_0^t V(s)\mathrm{d}s$.*

5.3. Regularity when $m = 0$ and $\lambda > 0$.

PROPOSITION 5.10. *Under the same hypotheses of Theorem* 4.8*, let V be the weak solution of* (63)*. Then, the velocity process $V(t) = \langle V, \delta_t\rangle$ is not well-defined, but the particle position process $X(t) = \langle V, 1_{[0,t]}\rangle$ is.*

Proof. We recall that the spectral density $\widehat{r}(\omega)$ from (67) satisfies $\lim_{\omega\to\infty}\widehat{r}(\omega) \in (0,\infty)$. This implies that $\widehat{r} \notin L^1(\mathbb{R})$. In view of Lemma 2.17, $V(t)$ is not well-defined since the spectral measure $\nu(\mathrm{d}\omega) = \widehat{r}(\omega)\mathrm{d}\omega$ is not finite. However, $X(t) = \langle V, 1_{[0,t]}\rangle$ is well-defined since $1_{[0,t]} \in L^2(\widehat{r})$. To show this, we invoke inequality (68) to estimate

$$\int_{\mathbb{R}} \left|\widehat{1_{[0,t]}}(\omega)\right|^2 \widehat{r}(\omega)\mathrm{d}\omega = 2\int_{\mathbb{R}} \frac{1-\cos(t\omega)}{\omega^2}\widehat{r}(\omega)\mathrm{d}\omega < \frac{2}{\pi\lambda}\int_{\mathbb{R}} \frac{1-\cos(t\omega)}{\omega^2}\mathrm{d}\omega < \infty. \qquad ∎$$

Since $V(t)$ is not well-defined, it is not certain whether $X(t)$ is differentiable. We are, however, able to assert the continuity of $X(t)$.

THEOREM 5.11. *Under the same hypotheses as in Theorem* 4.8, *$X(t)$ is a.s. continuous.*

Proof. In view of Proposition 3.18 in [9], it suffices to show that for fixed T, there exists $\kappa > 0$ such that for all $0 \leq s < t \leq T$,

$$\mathbb{E}\left|X(t) - X(s)\right|^2 \leq c_\kappa |t-s|^\kappa, \tag{81}$$

where $c_\kappa > 0$ is a constant. A straightforward calculation yields

$$\mathbb{E}\left|X(t) - X(s)\right|^2 = \int_{\mathbb{R}} \left|\widehat{1_{[s,t]}}(\omega)\right|^2 \widehat{r}(\omega)\mathrm{d}\omega = 4\int_0^\infty \frac{1-\cos\left((t-s)\omega\right)}{\omega^2}\widehat{r}(\omega)\mathrm{d}\omega. \tag{82}$$

Here we shall employ two elementary inequalities: for all $x \in \mathbb{R}$,

$$1 - \cos(x) \leq \frac{x^2}{2}, \tag{83}$$

and for every $\eta \in (0,1)$, there exists $c_\eta > 0$ such that for all x,

$$1 - \cos(x) \leq c_\eta x^\eta. \tag{84}$$

We estimate the last term of (82) using (84) with $\eta = 1/2$:

$$\begin{aligned}
&\int_0^\infty \frac{1-\cos\left((t-s)\omega\right)}{\omega^2}\widehat{r}(\omega)\mathrm{d}\omega \\
&\qquad = \int_0^1 \frac{1-\cos\left((t-s)\omega\right)}{\omega^2}\widehat{r}(\omega)\mathrm{d}\omega + \int_1^\infty \frac{1-\cos\left((t-s)\omega\right)}{\omega^2}\widehat{r}(\omega)\mathrm{d}\omega \\
&\qquad \leq |t-s|^2 \int_0^1 \widehat{r}(\omega)\mathrm{d}\omega + c_{1/2}|t-s|^{1/2}\int_1^\infty \frac{1}{\omega^{3/2}}\widehat{r}(\omega)\mathrm{d}\omega,
\end{aligned}$$

where in the last implication, we use (83) on the first term and (84) with $\eta = 1/2$ on the second term. We finally recall the fact that $\widehat{r}$ is bounded by $1/\pi\lambda$ from inequality (68) to obtain (81) with $\kappa = 1/2$. The proof is thus complete. ∎

5.4. Regularity when $m = 0$ and $\lambda = 0$. In this situation, once again $V(t)$ is not well-defined, but $X(t)$ is. We therefore are only able to investigate the continuity of $X(t)$. We begin by the following proposition.

PROPOSITION 5.12. *Under the same hypotheses as in Theorem* 4.10, *let V be the weak solution of* (63). *Then, $V(t) = \langle V, \delta_t\rangle$ is not well-defined, but $X(t) = \langle V, 1_{[0,t]}\rangle$ is.*

Proof. (a) $V(t)$ is not well-defined: in view of Lemma 2.17, it suffices to show that $\widehat{r}$ from (69) is not integrable, which implies that $\nu(\mathrm{d}\omega) = \widehat{r}(\omega)\mathrm{d}\omega$ is infinite. There are two cases.

If K satisfies (V), we write $\widehat{r}$ as

$$\frac{\pi\beta}{2}\widehat{r}(\omega) = \frac{\mathcal{K}_{\cos}(\omega)}{\mathcal{K}_{\cos}(\omega)^2 + \mathcal{K}_{\sin}(\omega)^2} = \frac{\omega^2\mathcal{K}_{\cos}(\omega)}{\left[\omega\mathcal{K}_{\cos}(\omega)\right]^2 + \left[\omega\mathcal{K}_{\sin}(\omega)\right]^2}. \tag{85}$$

It follows from (46) that $\lim_{\omega\to\infty}\left[\omega\mathcal{K}_{\cos}(\omega)\right]^2 + \left[\omega\mathcal{K}_{\sin}(\omega)\right]^2 = K(0)^2$. It remains to show that $\omega^2\mathcal{K}_{\cos}(\omega)$ is not integrable at infinity. Indeed, we recall from Lemma 3.4 that for all nonzero ω, $\omega^2\mathcal{K}_{\cos}(\omega) = \int_0^\infty K''(t)\left(1-\cos(t\omega)\right)\mathrm{d}t$. Since for all t, $K''(t)$ is not identical to zero and $K''(t)$ is continuous, we assume that there exists an interval (ϵ_1, ϵ_2) such that $K''(t) > 0$ for $t \in (\epsilon_1, \epsilon_2)$. We now integrate with respect to ω to find

$$\int_A^\infty \omega^2 \mathcal{K}_{\cos}(\omega)\mathrm{d}\omega = \int_A^\infty \int_0^\infty K''(t)\left(1-\cos(t\omega)\right)\mathrm{d}t\mathrm{d}\omega$$
$$\geq \int_{\epsilon_1}^{\epsilon_2} K''(t) \int_A^\infty \left(1-\cos(t\omega)\right)\mathrm{d}\omega\mathrm{d}t = \infty,$$

since for all $t \in (\epsilon_1, \epsilon_2)$, it is clear that $\int_A^\infty (1-\cos(t\omega))\,\mathrm{d}\omega = \infty$.

If K satisfies (VI), we observe that

$$\frac{\pi\beta}{2}\widehat{r}(\omega) = \frac{\mathcal{K}_{\cos}(\omega)}{\mathcal{K}_{\cos}(\omega)^2 + \mathcal{K}_{\sin}(\omega)^2} = \frac{\omega^{1-\sigma_2}\mathcal{K}_{\cos}(\omega)}{\left[\omega^{1-\sigma_2}\mathcal{K}_{\cos}(\omega)\right]^2 + \left[\omega^{1-\sigma_2}\mathcal{K}_{\sin}(\omega)\right]^2}\omega^{1-\sigma_2}, \tag{86}$$

where σ_2 is the constant from (VI). We invoke (47) to find that $\widehat{r}(\omega) \sim \omega^{1-\sigma_2}$ as $\omega \to \infty$.

We therefore conclude from both cases that $\widehat{r}(\omega) \notin L^1$.

(b) $X(t)$ is well-defined: this will follow immediately from Definition 2.16 if we can show that

$$\int_{\mathbb{R}} \left|\mathcal{F}\left[1_{[0,t]}\right](\omega)\right|^2 \widehat{r}(\omega)\mathrm{d}\omega < \infty, \tag{87}$$

which is equivalent to

$$\int_{\mathbb{R}} \frac{1-\cos(t\omega)}{\omega^2} \times \frac{\mathcal{K}_{\cos}(\omega)}{\mathcal{K}_{\cos}(\omega)^2 + \mathcal{K}_{\sin}(\omega)^2}\mathrm{d}\omega < \infty, \tag{88}$$

since $\mathcal{F}\left[1_{[0,t]}\right](\omega) = \widehat{1_{[0,t]}}(\omega) = \frac{1-e^{-it\omega}}{i\omega}$. When ω is near the origin, we recall from (71) that $\widehat{r}$ is always bounded regardless of the integrability of $K(t)$. The only concern is when ω tends to infinity. To this end, we employ (72) to infer for all ω sufficiently large that

$$\frac{1-\cos(t\omega)}{\omega^2} \times \frac{\mathcal{K}_{\cos}(\omega)}{\mathcal{K}_{\cos}(\omega)^2 + \mathcal{K}_{\sin}(\omega)^2} < c(\sigma)\frac{1-\cos(t\omega)}{\omega^{2-\sigma}}. \tag{89}$$

The RHS above is clearly integrable near infinity. We hence obtain (87). ☐

THEOREM 5.13. *Under the same hypotheses as in Theorem* 4.10*, let* $X(t)$ *be the particle position process from Proposition* 5.12*. Then,* $X(t)$ *is continuous a.s.*

Proof. To show continuity, we apply an argument similar to the one in the proof of Theorem 5.11. We recall from (84) that for the constant σ in (72), there exists $c_\sigma > 0$ such that for all $x \in \mathbb{R}$,

$$1 - \cos(x) \leq c_\sigma x^{(1-\sigma)/2}, \tag{90}$$

and $1 - \cos(x) \leq x^2/2$. We now fix A large enough such that for $\omega \geq A$, (72) holds. We then estimate for $t \neq s$ arbitrarily given,

$$\begin{aligned}
&\int_0^\infty \frac{1-\cos\left((t-s)\omega\right)}{\omega^2}\widehat{r}(\omega)\mathrm{d}\omega \\
&= \int_0^A \frac{1-\cos\left((t-s)\omega\right)}{\omega^2}\widehat{r}(\omega)\mathrm{d}\omega + \int_A^\infty \frac{1-\cos\left((t-s)\omega\right)}{\omega^2}\widehat{r}(\omega)\mathrm{d}\omega \\
&\leq |t-s|^2\int_0^A \widehat{r}(\omega)\mathrm{d}\omega + c_\sigma|t-s|^{(1-\sigma)/2}\int_1^\infty \frac{c}{\omega^{1+(1-\sigma)/2}}\mathrm{d}\omega,
\end{aligned} \tag{91}$$

where c, c_σ are from (72), (90) respectively. We thus obtain an estimate similar to (81); namely, for $0 \le s, t \le T$, there exists $C = C(T)$ such that

$$\mathbb{E}\left|X(t) - X(s)\right|^2 \le C|t-s|^{(1-\sigma)/2}. \tag{92}$$

The continuity of $X(t)$ follows immediately from Proposition 3.18 in [9], which concludes the proof. ∎

6. Asymptotic analysis of the mean-squared displacement. We are now prepared to prove our version of the meta-theorem (4) that was presented in the introduction. Having established basic properties of the spectral density $\hat{r}$ in the last two sections, the Abelian theorem for Fourier transforms from section 3 will allow us to immediately handle the case when $m > 0$ or $\lambda > 0$. As has been the case throughout the paper, $m = \lambda = 0$ presents a greater challenge and requires more restrictions on the memory $K(t)$.

Throughout this section, let $X(t)$ be the GLE position process as defined by Definition 2.16.

6.1. Asymptotic of the MSD when either $m > 0$ or $\lambda > 0$.

THEOREM 6.1. *Suppose that either $m > 0$ or $\lambda > 0$, and assume that K satisfies* (I) + (II). *Then*

$$\lim_{t\to\infty} \frac{\mathbb{E}\left[X^2(t)\right]}{t} = C \in (0,\infty);$$

i.e., the process $X(t)$ is asymptotically diffusive.

Proof. Using definition (13), we have

$$\mathbb{E}\left[X^2(t)\right] = \int_{\mathbb{R}} \left|\widehat{1_{[0,t]}}(\omega)\right|^2 \widehat{r}(\omega)\mathrm{d}\omega = 2\int_{\mathbb{R}} \frac{1-\cos(t\omega)}{\omega^2}\widehat{r}(\omega)\mathrm{d}\omega. \tag{93}$$

By changing variable $z := t\omega$, we obtain

$$\mathbb{E}\left[X^2(t)\right] = 2t\int_{\mathbb{R}} \frac{1-\cos(z)}{z^2}\widehat{r}\left(\frac{z}{t}\right)\mathrm{d}z,$$

which implies

$$\frac{\mathbb{E}\left[X^2(t)\right]}{t} = 2\int_{\mathbb{R}} \frac{1-\cos(z)}{z^2}\widehat{r}\left(\frac{z}{t}\right)\mathrm{d}z. \tag{94}$$

We remind the reader that by (66), the general form of $\widehat{r}(\omega)$ is

$$\widehat{r}(\omega) = \frac{2\lambda + \beta\widehat{K}(\omega)}{2\pi\left|im\omega + \lambda + \beta\widehat{K^+}(\omega)\right|^2}.$$

Since K is integrable by condition (II), either $m > 0$ or $\lambda > 0$ implies that $\lim_{\omega\to 0}\widehat{r}(\omega) = \widehat{r}(0) \in (0,\infty)$. In addition, by condition (I), Lemma 5.2 implies that $\widehat{r}(\omega)$ is bounded at infinity. As a consequence, $\widehat{r}(\omega)$ is bounded in $\mathbb{R}$. By the dominated convergence theorem, we obtain

$$\lim_{t\to\infty} \frac{\mathbb{E}\left[X^2(t)\right]}{t} = 2\int_{\mathbb{R}} \frac{1-\cos(z)}{z^2}\widehat{r}(0)\mathrm{d}z \in (0,\infty).$$ ∎

THEOREM 6.2. *Suppose that either $m > 0$ or $\lambda > 0$, and assume that K satisfies* (I) + (III). *Then*

$$\lim_{t\to\infty} \frac{\mathbb{E}\left[X^2(t)\right]}{t^\alpha} = C \in (0,\infty),$$

where α is the constant from condition (III).

Proof. From (94), we have

$$\frac{\mathbb{E}\left[X^2(t)\right]}{t^\alpha} = 2\int_{\mathbb{R}} \frac{1-\cos(z)}{z^{1+\alpha}} \times \frac{\widehat{r}(\frac{z}{t})}{(\frac{z}{t})^{1-\alpha}} \mathrm{d}z.$$

We observe that (66) is equivalent to

$$\frac{\widehat{r}(\omega)}{\omega^{1-\alpha}} = \frac{1}{\pi} \times \frac{\lambda\omega^{1-\alpha} + \beta\omega^{1-\alpha}\mathcal{K}_{\cos}(\omega)}{\left[\lambda\omega^{1-\alpha} + \beta\omega^{1-\alpha}\mathcal{K}_{\cos}(\omega)\right]^2 + \left[m\omega^{2-\alpha} - \beta\omega^{1-\alpha}\mathcal{K}_{\sin}(\omega)\right]^2}.$$

Proposition 3.1 implies that

$$\lim_{\omega\to 0} \frac{\widehat{r}(\omega)}{\omega^{1-\alpha}} = c \in (0,\infty),$$

and subsequently, $\widehat{r}(\omega)/\omega^{1-\alpha}$ is bounded on $(0,\infty)$ since by Lemma 5.2, $\widehat{r}(\omega)$ is bounded at infinity. Applying the dominated convergence theorem gives

$$\lim_{t\to\infty} \frac{\mathbb{E}\left[X^2(t)\right]}{t^\alpha} = 2c\int_{\mathbb{R}} \frac{1-\cos(z)}{z^{1+\alpha}} \mathrm{d}z \in (0,\infty).$$

The proof is complete. ∎

6.2. Asymptotics of the MSD when $m = \lambda = 0$.

THEOREM 6.3. *Suppose that $m = \lambda = 0$ and that K satisfies Assumption* 1.3 + (I). *Then,*

(a) *if K satisfies* (II),

$$\lim_{t\to\infty} \frac{\mathbb{E}\left[X^2(t)\right]}{t} = C \in (0,\infty);$$

(b) *if K satisfies* (III),

$$\lim_{t\to\infty} \frac{\mathbb{E}\left[X^2(t)\right]}{t^\alpha} = C \in (0,\infty),$$

where α is the constant from condition (III).

Proof. (a) We recall from (94) that

$$\frac{\mathbb{E}\left[X^2(t)\right]}{t} = 2\int_{\mathbb{R}} \frac{1-\cos(z)}{z^2} \widehat{r}\left(\frac{z}{t}\right) \mathrm{d}z. \tag{95}$$

It therefore suffices to show that

$$\lim_{t\to\infty} \int_0^\infty \frac{1-\cos(z)}{z^2} \widehat{r}\left(\frac{z}{t}\right) \mathrm{d}z = C \in (0,\infty). \tag{96}$$

Fixing A such that for all $\omega \geq A$, (72) holds, we decompose the integral above as

$$\int_0^\infty \frac{1-\cos(z)}{z^2} \widehat{r}\left(\frac{z}{t}\right) \mathrm{d}z = \int_0^{At} + \int_{At}^\infty \frac{1-\cos(z)}{z^2} \widehat{r}\left(\frac{z}{t}\right) \mathrm{d}z = I_5(t) + I_6(t). \tag{97}$$

We claim that $I_6(t) \to 0$ as $t \to \infty$. Indeed, since $z \geq At$, by (72), $\widehat{r}(z/t) \leq c(z/t)^\sigma$. We have a chain of implications

$$I_6(t) \leq \int_{At}^{\infty} \frac{1-\cos(z)}{z^2} \times c\left(\frac{z}{t}\right)^\sigma \mathrm{d}z = \frac{c}{t^\sigma}\int_{At}^{\infty}\frac{1-\cos(z)}{z^{2-\sigma}} \to 0. \tag{98}$$

We write $I_5(t)$ as

$$I_5(t) = \int_0^\infty 1_{(0,At]}(z)\frac{1-\cos(z)}{z^2}\widehat{r}\left(\frac{z}{t}\right)\mathrm{d}z. \tag{99}$$

Since $\widehat{r}(\omega)$ is bounded on $(0,A]$ and $z/t \leq A$, the integrand above is dominated by $\frac{1-\cos(z)}{z^2}$, which is integrable. It follows from the dominated convergence theorem that

$$\lim_{t\to\infty} I_5(t) = \widehat{r}(0)\int_0^\infty \frac{1-\cos(z)}{z^2}\mathrm{d}z \in (0,\infty). \tag{100}$$

We finally combine (98) and (100) to obtain (96), which concludes part (a).

(b) First, in view of Proposition 3.1, since $K(t)$ satisfies (III), we have that

$$\lim_{\omega\to 0}\omega^{1-\alpha}\widehat{r}(\omega) = \frac{2}{\pi\beta}\lim_{\omega\to 0}\omega^{1-\alpha}\frac{\mathcal{K}_{\cos}(\omega)}{\mathcal{K}_{\cos}(\omega)^2+\mathcal{K}_{\sin}(\omega)^2} = C \in (0,\infty). \tag{101}$$

Since

$$\frac{\mathbb{E}\left[X^2(t)\right]}{t^\alpha} = 2t^{1-\alpha}\int_{\mathbb{R}}\frac{1-\cos(z)}{z^2}\widehat{r}\left(\frac{z}{t}\right)\mathrm{d}z, \tag{102}$$

it suffices to show that

$$\lim_{t\to\infty} t^{1-\alpha}\int_0^\infty \frac{1-\cos(z)}{z^2}\widehat{r}\left(\frac{z}{t}\right)\mathrm{d}z = C \in (0,\infty). \tag{103}$$

Fixing the same A from part (a), we have

$$\int_0^\infty t^{1-\alpha}\frac{1-\cos(z)}{z^2}\widehat{r}\left(\frac{z}{t}\right)\mathrm{d}z = \int_0^{At} + \int_{At}^\infty t^{1-\alpha}\frac{1-\cos(z)}{z^2}\widehat{r}\left(\frac{z}{t}\right)\mathrm{d}z = I_7(t)+I_8(t). \tag{104}$$

To show that $I_8(t) \to 0$ as $t\to\infty$, we have a chain of implications

$$\begin{aligned} I_8(t) &\leq t^{1-\alpha}\int_{At}^\infty c\,\frac{1-\cos(z)}{z^2}\left(\frac{z}{t}\right)^\sigma \mathrm{d}z = \frac{ct^{1-\alpha}}{t^\sigma}\int_{At}^\infty\frac{1-\cos(z)}{z^{2-\sigma}}\mathrm{d}z \\ &\leq \frac{ct^{1-\alpha}}{t^\sigma}\times\frac{1}{A^{1-\sigma}t^{1-\sigma}} = \frac{c}{t^\alpha} \to 0, \end{aligned} \tag{105}$$

where the constant c may change from line to line independent of t. Next, we write $I_7(t)$ as

$$I_7(t) = \int_0^\infty 1_{(0,At]}(z)\frac{1-\cos(z)}{z^{1+\alpha}}\left(\frac{z}{t}\right)^{1-\alpha}\widehat{r}\left(\frac{z}{t}\right)\mathrm{d}z. \tag{106}$$

From (101), we observe that $\omega^{1-\alpha}\widehat{r}(\omega)$ is bounded on $(0,A]$. Since $z/t \leq A$, the integrand above is dominated by $\frac{1-\cos(z)}{z^{1+\alpha}}$, which is integrable. Taking t to infinity, it follows from the dominated convergence theorem that

$$\lim_{t\to\infty} I_7(t) = \int_0^\infty\frac{1-\cos(z)}{z^{1+\alpha}}\mathrm{d}z\lim_{\omega\to 0}\omega^{1-\alpha}\widehat{r}(\omega) \in (0,\infty). \tag{107}$$

Finally, (103) follows from (105) and (107). The proof is thus complete. ∎

7. Transient anomalous diffusion. As mentioned when we introduced the generalized Rouse family of memory kernels in section 2.6.1, a sum of exponentials can be used to approximate a power law. This is an appealing property because, for such memory kernels, the non-Markov GLE can be rewritten as a high-dimensional system of SDEs. (See [8] or [30] for discussions of the finite-dimensional case.) Since a finite sum of exponentials will always be integrable, the associated solutions to the GLE will be asymptotically diffusive. Nevertheless, the MSD of these solutions will look subdiffusive over a large time range if the memory kernel has an appropriate form.

In this section, we propose a rigorous definition of *transient anomalous diffusion* in the case where either $m > 0$ or $\lambda > 0$. We formulate the result in such a way that one can check a convergence condition on the sequence of memory kernels and then have that for any interval $[0, T]$, there is an N sufficiently large so that the GLE with N terms is arbitrarily close to the limiting MSD over $[0, T]$. One might think that such a result is automatic, but the argument is more subtle than expected. We provide some results in this direction. Once again, the analysis is more subtle when $m > 0$ and $\lambda = 0$ (Theorem 7.1) and easier when $\lambda > 0$ (Theorem 7.5). However, we do not have a result of this kind for $m = \lambda = 0$.

THEOREM 7.1. *Suppose that $m > 0$ and $\lambda = 0$. Assume all of the following:*
(a) $K_n \in C^2(0,\infty)$ *satisfies* (I) + (II).
(b) $K \in C^2(0,\infty)$ *satisfies* (I) + (III).
(c) *For all $n \in \mathbb{N}$, $K_n(t)$ is convex on $(0,\infty)$.*
(d) *As $n \to \infty$, $K_n(t) \longrightarrow K(t)$ for all $t > 0$.*
(e) *There exists a constant $\kappa \in (0,1)$ such that*

$$\sup_{n\in\mathbb{N}^+} \sup_{t\in(0,1]} t^\kappa K_n(t) < \infty. \tag{108}$$

Let X_n, X be the particle position processes as in Corollary 5.5 *associated with K_n, K, respectively. Then for all $T > 0$,*

$$\lim_{n\to\infty} \left[\sup_{t,s\in[0,T]} \left| \mathbb{E}\left[X_n(t)X_n(s)\right] - \mathbb{E}\left[X(t)X(s)\right] \right| \right] = 0. \tag{109}$$

In order to prove Theorem 7.1, we need some preliminary facts.

LEMMA 7.2. *For $x, y \in \mathbb{R}$, there holds*

$$|\cos(x-y) - \cos(x) - \cos(y) + 1| \le 2 - \cos(x) - \cos(y).$$

Proof. Our inequality is equivalent to

$$-2 + \cos(x) + \cos(y) \le \cos(x-y) - \cos(x) - \cos(y) + 1 \le 2 - \cos(x) - \cos(y).$$

The RHS inequality is evident. We are left to prove

$$-2 + \cos(x) + \cos(y) \le \cos(x-y) - \cos(x) - \cos(y) + 1,$$

which can be written as

$$2\left[\sin^2(x/2) + \sin^2(y/2) + 2\cos(x/2)\cos(y/2)\sin(x/2)\sin(y/2)\right] \\ + (1-\cos(x))(1-\cos(y)) \ge 0,$$

which in turn always holds. ∎

We now assert that the Fourier cosine and sine transforms of K_n converge pointwise to those of K.

PROPOSITION 7.3. *Suppose that* $\{K_n\}_{n\geq 1}$ *and* K *satisfy* (I) *and that for every* $t>0$, $K_n(t)\to K(t)$ *as* $n\to\infty$. *For each* $n\geq 1$, *put*

$$\mathcal{K}^n_{\cos}(\omega)=\int_0^\infty K_n(t)\cos(t\omega)\mathrm{d}t, \qquad \mathcal{K}^n_{\sin}(\omega)=\int_0^\infty K_n(t)\sin(t\omega)\mathrm{d}t.$$

Then, for nonzero ω,

$$\lim_{k\to\infty}\mathcal{K}^n_{\cos}(\omega)=\mathcal{K}_{\cos}(\omega) \quad \textit{and} \quad \lim_{k\to\infty}\mathcal{K}^n_{\sin}(\omega)=\mathcal{K}_{\sin}(\omega). \tag{110}$$

Proof. Given $\varepsilon>0$, fix A large enough such that

$$\left|\int_0^\infty K(t)\cos(t\omega)\mathrm{d}t-\int_0^A K(t)\cos(t\omega)\mathrm{d}t\right|<\varepsilon \quad \text{and} \quad \frac{8K(A)}{\omega}<\varepsilon, \tag{111}$$

where the latter condition is possible since $K(t)$ eventually decreases to 0 as $t\to\infty$. We have

$$\int_0^A K_n(t)\cos(t\omega)\mathrm{d}t \overset{n\to\infty}{\longrightarrow} \int_0^A f(t)\cos(t\omega)\mathrm{d}t, \tag{112}$$

by virtue of the dominated convergence theorem. For each n, we note that inequality (16) also holds for K_n, which implies

$$\left|\int_A^\infty K_n(t)\cos(t\omega)\mathrm{d}t\right|\leq\frac{4K_n(A)}{\omega}\leq\frac{8K(A)}{\omega}<\varepsilon, \tag{113}$$

since $K_n(A)\to K(A)$ as $n\to\infty$. It follows immediately that

$$\left|\int_0^\infty K_n(t)\cos(t\omega)\mathrm{d}t-\int_0^A K_n(t)\cos(t\omega)\mathrm{d}t\right|<\varepsilon,$$

which is equivalent to

$$\int_0^A K_n(t)\cos(t\omega)\mathrm{d}t-\varepsilon<\int_0^\infty K_n(t)\cos(t\omega)\mathrm{d}t<\int_0^A K_n(t)\cos(t\omega)\mathrm{d}t+\varepsilon. \tag{114}$$

We now send n to infinity and combine (111), (112), and (114) to obtain the Fourier cosine limit in (110). We then apply a similar argument to establish the Fourier sine limit. ∎

As a direct consequence of Proposition 7.3, we obtain uniform bounds on $\{\mathcal{K}^n_{\cos}\}_{n\geq 1}$ and $\{\mathcal{K}^n_{\sin}\}_{n\geq 1}$ in the following lemma.

LEMMA 7.4. *Let* K_n, K *be as in Theorem* 7.1. *Then for every* $\omega_0>1$, *there exists* $N>0$ *sufficiently large such that*

$$\inf_{n\geq N}\inf_{\omega\in(0,\omega_0]}\mathcal{K}^n_{\cos}(\omega)>0, \tag{115}$$

and

$$\sup_{n\geq N}\sup_{\omega>\omega_0}\mathcal{K}^n_{\cos}(\omega)<\infty, \quad \sup_{n\geq N}\sup_{\omega>\omega_0}\mathcal{K}^n_{\sin}(\omega)<\infty. \tag{116}$$

Proof. We first note that $\{K_n\}_{n\geq 1}$ are convex, and so is K, being the limiting function. Furthermore, convexity and eventually decreasing to zero imply that $K_n(t)$ is actually decreasing to zero for $t \in [0,\infty)$.

We now invoke (108) to see that $\lim_{t\to 0} tK_n(t) = 0$. In view of Lemma 3.4, $\mathcal{K}^n_{\cos}$ satisfies formula (41). We then estimate

$$\int_0^\infty K_n(t)\cos(t\omega)\mathrm{d}t = \frac{1}{\omega^2}\int_0^\infty K_n''(t)(1-\cos(t\omega))\mathrm{d}t \geq \int_0^{t_1} K_n''(t)\frac{1-\cos(t\omega)}{\omega^2}\mathrm{d}t = \int_0^{t_1} t^2 K_n''(t)\frac{1-\cos(t\omega)}{(t\omega)^2}\mathrm{d}t.$$

Observe that $\min_{x\in(0,\pi/2]}\frac{1-\cos(x)}{x^2} = c_1 > 0$. Fixing $\omega_0 > 1$ and setting $t_1 = \pi/(2\omega_0)$, for $\omega \in (0,\omega_0]$, we have

$$\frac{1}{\omega^2}\int_0^\infty K_n''(t)(1-\cos(t\omega))\mathrm{d}t \geq c_1\int_0^{t_1} t^2 K_n''(t)\mathrm{d}t. \tag{117}$$

Integrating by parts, the above RHS yields

$$\int_0^{t_1} t^2 K_n''(t)\mathrm{d}t = t_1^2 K_n'(t_1) - t_1 K_n(t_1) + \int_0^{t_1} K_n(t)\mathrm{d}t.$$

Fix $0 < t^* < t_1$ to be chosen later. The mean value theorem implies

$$\frac{K_n(t^*) - K_n(t_1)}{t^* - t_1} = K_n'(\xi) \qquad (t^* < \xi < t_1), \\ \leq K_n'(t_1),$$

since $K_n'(t)$ is increasing on $t \in (0,\infty)$. It follows that

$$\int_0^{t_1} t^2 K_n''(t)\mathrm{d}t \geq t_1^2\frac{K_n(t^*) - K_n(t_1)}{t^* - t_1} - t_1 K_n(t_1) + \int_0^{t_1} K_n(t)\mathrm{d}t.$$

Letting $n \to \infty$, we obtain

$$\begin{aligned}\liminf_{n\to\infty}\int_0^{t_1} t^2 K_n''(t)\mathrm{d}t &\geq t_1^2\frac{K(t^*) - K(t_1)}{t^* - t_1} - t_1 K(t_1) + \int_0^{t_1} K(t)\mathrm{d}t \\ &= t_1^2\left[\frac{K(t^*) - K(t_1)}{t^* - t_1} - K'(t_1)\right] + t_1^2 K'(t_1) - t_1 K(t_1) + \int_0^{t_1} K(t)\mathrm{d}t \\ &= t_1^2\left[\frac{K(t^*) - K(t_1)}{t^* - t_1} - K'(t_1)\right] + \int_0^{t_1} t^2 K''(t)\mathrm{d}t.\end{aligned}$$

As $t^* \to t_1$, on the RHS above, the bracket tends to 0, whereas the integral is positive. Subsequently, we can choose t^* close enough to t_1 such that the RHS above is positive. Thus,

$$\liminf_{n\to\infty}\int_0^{t_1} t^2 K_n''(t)\mathrm{d}t > 0. \tag{118}$$

We then combine (118) with (117) to infer the existence of a constant $c_2 = c_2(\omega_0) > 0$ such that for n large and $\omega \in (0, \omega_0]$, it holds that

$$\int_0^\infty K_n(t)\cos(t\omega)\mathrm{d}t \geq c_2 > 0, \tag{119}$$

which proves (115). Now for $\omega > \omega_0$, by a change of variable, we have

$$\begin{aligned}\int_0^\infty K_n(t)\cos(t\omega)\mathrm{d}t &= \frac{1}{\omega}\int_0^\infty K_n\left(\frac{z}{\omega}\right)\cos(z)\mathrm{d}z \\ &= \frac{1}{\omega^{1-\kappa}}\int_0^1 \left(\frac{z}{\omega}\right)^\kappa K_n\left(\frac{z}{\omega}\right)\frac{\cos(z)}{z^\kappa}\mathrm{d}z + \frac{1}{\omega}\int_1^\infty K_n\left(\frac{z}{\omega}\right)\cos(z)\mathrm{d}z.\end{aligned}$$

To estimate the first integral on the above RHS, we employ the fact that $t^\kappa K_n(t)$ is uniformly bounded on $(0,1]$ from (108) to find

$$\begin{aligned}\frac{1}{\omega^{1-\kappa}}\int_0^1 \left(\frac{z}{\omega}\right)^\kappa K_n\left(\frac{z}{\omega}\right)\frac{\cos(z)}{z^\kappa}\mathrm{d}z &\leq \frac{\sup_{n\in\mathbb{N}^+}\sup_{t\in(0,1]} t^\kappa K_n(t)}{\omega^{1-\kappa}}\int_0^1 \frac{\cos(z)}{z^\kappa}\mathrm{d}z \\ &\leq \sup_{n\in\mathbb{N}^+}\sup_{t\in(0,1]} t^\kappa K_n(t)\int_0^1\frac{\cos(z)}{z^\kappa}\mathrm{d}z,\end{aligned}$$

where the last implication simply follows from the assumption $\omega > \omega_0 > 1$. For the other integral, invoking the second mean value theorem again gives

$$\frac{1}{\omega}\int_1^\infty K_n\left(\frac{z}{\omega}\right)\cos(z)\mathrm{d}z \leq \frac{2}{\omega}K_n\left(\frac{1}{\omega}\right) \leq 2\sup_{n\in\mathbb{N}^+}\sup_{t\in(0,1]} tK_n(t) \leq 2\sup_{n\in\mathbb{N}^+}\sup_{t\in(0,1]} t^\kappa K_n(t),$$

since $\kappa \in (0,1)$ by assumption (d) of Theorem 7.1. We thus obtain for every $\omega \geq \omega_0$ and $n > 0$

$$\int_0^\infty K_n(t)\cos(t\omega)\mathrm{d}t \leq \left(2 + \int_0^1 \frac{\cos(z)}{z^\kappa}\mathrm{d}z\right)\sup_{n\in\mathbb{N}^+}\sup_{t\in(0,1]} t^\kappa K_n(t) \tag{120}$$

and likewise

$$\int_0^\infty K_n(t)\sin(t\omega)\mathrm{d}t \leq \left(2 + \int_0^1 \frac{\cos(z)}{z^\kappa}\mathrm{d}z\right)\sup_{n\in\mathbb{N}^+}\sup_{t\in(0,1]} t^\kappa K_n(t), \tag{121}$$

which proves (116). The proof is thus complete. □

We are now ready to give the proof of the main theorem in this section.

Proof of Theorem 7.1. A short computation yields

$$\mathbb{E}\left[X_n(t)X_n(s)\right] = \int_{\mathbb{R}} \frac{\cos((t-s)\omega) - \cos(t\omega) - \cos(s\omega) + 1}{\omega^2}\,\widehat{r}_n(\omega)\mathrm{d}\omega. \tag{122}$$

For $t, s \in [0, T]$, we estimate

$$\begin{aligned}&|\mathbb{E}\left[X_n(t)X_n(s)\right] - \mathbb{E}\left[X(t)X(s)\right]| \\ &\qquad\leq \int_{\mathbb{R}} \frac{|\cos((t-s)\omega) - \cos(t\omega) - \cos(s\omega) + 1|}{\omega^2}\left|\widehat{r_n}(\omega) - \widehat{r}(\omega)\right|\mathrm{d}\omega \\ &\qquad\leq \int_{\mathbb{R}} \frac{2 - \cos(t\omega) - \cos(s\omega)}{\omega^2}\left|\widehat{r_n}(\omega) - \widehat{r}(\omega)\right|\mathrm{d}\omega \\ &\qquad= \int_{\mathbb{R}} \frac{1-\cos(t\omega)}{\omega^2}|\widehat{r_n}(\omega) - \widehat{r}(\omega)|\mathrm{d}\omega + \int_{\mathbb{R}} \frac{1-\cos(s\omega)}{\omega^2}|\widehat{r_n}(\omega) - \widehat{r}(\omega)|\mathrm{d}\omega.\end{aligned}$$

It follows that

(123)
$$\sup_{t,s\in[0,T]} |\mathbb{E}[X_n(t)X_n(s)] - \mathbb{E}[X(t)X(s)]| \le 2 \sup_{t\in[0,T]} \int_{\mathbb{R}} \frac{1-\cos(t\omega)}{\omega^2} |\widehat{r_n}(\omega) - \widehat{r}(\omega)| \mathrm{d}\omega.$$

We note that for $0 \le t \le T$,

$$\frac{1-\cos(t\omega)}{\omega^2} = t^2 \frac{1-\cos(t\omega)}{(t\omega)^2} \le T^2 \sup_{\omega\in\mathbb{R}} \frac{1-\cos(\omega)}{\omega^2}.$$

We then combine with inequality (123) to see that

$$\begin{aligned}(124)\quad & \sup_{t,s\in[0,T]} |\mathbb{E}[X_n(t)X_n(s)] - \mathbb{E}[X(t)X(s)]| \\ & \qquad \le 2T^2 \int_{\mathbb{R}} |\widehat{r}_n(\omega) - \widehat{r}(\omega)| \mathrm{d}\omega \sup_{\omega\in\mathbb{R}\setminus\{0\}} \frac{1-\cos(\omega)}{\omega^2}.\end{aligned}$$

The problem now is reduced to showing that $\widehat{r}_n \to \widehat{r}$ in $L^1(\mathbb{R})$. In view of Proposition 7.3, for $\omega > 0$, $\widehat{r}_n(\omega) = \widehat{r}(\omega)$ as $n \to \infty$. It remains to find a dominating function. Let ω_0 be the constant from Lemma 7.4. There are two cases. On one hand, if $\omega \le \omega_0$, recalling formula (65), we have

$$\begin{aligned}\pi\widehat{r}_n(\omega) &= \frac{2\beta\int_0^\infty K_n(t)\cos(t\omega)\mathrm{d}t}{\left[\beta\int_0^\infty K_n(t)\cos(t\omega)\mathrm{d}t\right]^2 + \left[m\omega - \beta\int_0^\infty K_n(t)\sin(t\omega)\mathrm{d}t\right]^2} \\ &\le \frac{2}{\beta\int_0^\infty K_n(t)\cos(t\omega)\mathrm{d}t} \le \frac{2}{c_2},\end{aligned}$$

where $c_2 = c_2(\omega_0)$ is the constant in (119). On the other hand, observe that the constant $c_3 := \left(2 + \int_0^1 \frac{\cos(z)}{z^\kappa}\mathrm{d}z\right) \sup_{n\in\mathbb{N}^+} \sup_{t\in(0,1]} t^\kappa K_n(t)$ in (120) and (121) does not depend on ω_0. We thus can choose ω_0 large enough such that $c_3/\omega_0 < m$. Hence, for $\omega > \omega_0$, we have

$$\pi\widehat{r}_n(\omega) \le \frac{2\beta\int_0^\infty K_n(t)\cos(t\omega)\mathrm{d}t}{\left[m\omega - \beta\int_0^\infty K_n(t)\sin(t\omega)\mathrm{d}t\right]^2} \le \frac{2c_3}{\omega^2\left[m - c_3/\omega_0\right]^2},$$

where the last implication follows from (120) and (121). Combining the two cases above, we infer the following function:

$$g(\omega) = \frac{2}{c_2} 1_{(0,\omega_0]}(\omega) + \frac{2c_3}{\omega^2\left[m - c_3/\omega_0\right]^2} 1_{(\omega_0,\infty)}(\omega),$$

dominating $\widehat{r}_n(\omega)$ in $\mathbb{R}$. It is also clear that $g \in L^1(\mathbb{R})$. The dominated convergence theorem then implies that $\widehat{r_n}$ converges to $\widehat{r}$ in $L^1(\mathbb{R})$. As a consequence, we obtain (109), which follows from (124). The proof is thus complete. ∎

We finally assert a result similar to Theorem 7.1 for the case ($m \ge 0$, $\lambda > 0$), in which minimal assumptions on memory kernels are required.

THEOREM 7.5. *Suppose that $m \ge 0$ and $\lambda > 0$. Assume all of the following:*
(a) K_n *satisfies* (I) + (II).
(b) K *satisfies* (I) + (III).

(c) *As* $n \to \infty$, $K_n(t) \longrightarrow K(t)$ *for all* $t > 0$.
Let $X_n(t) = \langle V_n, 1_{[0,t]} \rangle$, $X(t) = \langle V, 1_{[0,t]} \rangle$ *be the particle position processes as in Definition* 2.16*, where* V_n, V *are, as in either Theorem* 4.5 *(when* $m > 0$, $\lambda > 0$*) or Theorem* 4.8 *(when* $m = 0$, $\lambda > 0$*), associated with* K_n, K*, respectively. Then for all* $T > 0$,

$$\lim_{n\to\infty} \left[\sup_{t,s\in[0,T]} \left| \mathbb{E}\left[X_n(t)X_n(s)\right] - \mathbb{E}\left[X(t)X(s)\right] \right| \right] = 0. \tag{125}$$

Proof. Let $\widehat{r}_n$, $\widehat{r}$ be spectral densities associated with X_n and X, respectively. Note that inequality (123) is still valid regardless of λ:

$$\sup_{t,s\in[0,T]} \left|\mathbb{E}\left[X_n(t)X_n(s)\right] - \mathbb{E}\left[X(t)X(s)\right]\right| \le 2 \sup_{t\in[0,T]} \int_{\mathbb{R}} \frac{1-\cos(t\omega)}{\omega^2} |\widehat{r_n}(\omega) - \widehat{r}(\omega)| \mathrm{d}\omega. \tag{126}$$

Now, for $0 \le t \le T$, we have

$$\begin{aligned} &\int_{\mathbb{R}} \frac{1-\cos(t\omega)}{\omega^2} |\widehat{r}_n(\omega) - \widehat{r}(\omega)| \mathrm{d}\omega \\ &\quad = \int_{|\omega|\le 1} \frac{1-\cos(t\omega)}{\omega^2} |\widehat{r}_n(\omega) - \widehat{r}(\omega)| \mathrm{d}\omega + \int_{|\omega|>1} \frac{1-\cos(t\omega)}{\omega^2} |\widehat{r}_n(\omega) - \widehat{r}(\omega)| \mathrm{d}\omega \\ &\quad \le T^2 \sup_{y\in\mathbb{R}} \frac{1-\cos(y)}{y^2} \int_{|\omega|\le 1} |\widehat{r}_n(\omega) - \widehat{r}(\omega)|\, \mathrm{d}\omega + \int_{|\omega|>1} \frac{2}{\omega^2} |\widehat{r}_n(\omega) - \widehat{r}(\omega)| \mathrm{d}\omega. \end{aligned} \tag{127}$$

Recall from (66) that $\widehat{r_n}$ is given by

$$\pi \widehat{r}_n(\omega) = \frac{\lambda + \beta \int_0^\infty K_n(t)\cos(t\omega)\mathrm{d}t}{\left[\lambda + \beta \int_0^\infty K_n(t)\cos(t\omega)\mathrm{d}t\right]^2 + \left[m\omega - \beta \int_0^\infty K_n(t)\sin(t\omega)\mathrm{d}t\right]^2},$$

which immediately yields the estimate $\widehat{r}_n(\omega) \le \frac{1}{\pi\lambda}$, and likewise, $\widehat{r}(\omega) \le \frac{1}{\pi\lambda}$. In addition, since K_n converges to K pointwise, Proposition 7.3 implies that $\widehat{r_n}(\omega)$ converges to $\widehat{r}(\omega)$ for every nonzero ω. It follows from the dominated convergence theorem that

$$\lim_{n\to\infty} \int_{|\omega|\le 1} |\widehat{r}_n(\omega) - \widehat{r}(\omega)|\, \mathrm{d}\omega = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_{|\omega|>1} \frac{2}{\omega^2} |\widehat{r}_n(\omega) - \widehat{r}(\omega)| \mathrm{d}\omega = 0. \tag{128}$$

We finally combine (126), (127), and (128) to deduce the desired limit (125), which concludes the proof. ∎

8. Discussion. We have established rigorous results for two important aspects of solutions to the generalized Langevin equation (GLE): (1) well-posedness and regularity for a wide class of memory kernels, and (2) a direct connection between the tail behavior of the memory kernel K and the large-time growth of the particle's mean-squared displacement (MSD). In contrast to intuition that was based on self-similar subdiffusive processes [24], we showed that regularity and large-time asymptotics are separate issues. For example, a fractional Brownian motion (FBM) with Hurst parameter $H \in (0, \frac{1}{2})$ is subdiffusive with MSD $\sim t^{2H}$ for large t and has sample paths that are C^γ-Hölder continuous only for $\gamma \in (0, H)$. For FBM, being more subdiffusive implies being less regular. As a result of our work, we now see it is possible to have a subdiffusive process that is differentiable. For example, we can consider the $N \to \infty$ limit of the generalized Rouse kernel (28) with $\nu > 1$ with $m > 0$.

Regarding subdiffusivity, we have provided a broad set of sufficient conditions on a memory kernel K for solutions of the associated GLE to be asymptotically diffusive or subdiffusive. Our findings are consistent with the long-held belief that if $K(t) \sim t^{-\alpha}$ as $t \to \infty$, then $\mathbb{E}\left[X^2(t)\right] \sim t^{\alpha}$ as $t \to \infty$ [28, 18]. There are at least two major barriers to finding conditions that are necessary and sufficient. These can be understood by looking at the three main steps of the proof. (In each of the heuristic statements below, take $\alpha \in (0,1)$.)

Step 1: *Relating the tail of K to near-zero behavior of $\mathcal{K}_{\cos}$ and $\mathcal{K}_{\sin}$*:

$$\big[K(t) \sim t^{-\alpha} \text{ as } t \to \infty\big] \Longleftrightarrow \big[\mathcal{K}_{\cos}(\omega), \mathcal{K}_{\sin}(\omega) \sim \omega^{\alpha-1} \text{ as } \omega \to 0\big].$$

If either $m > 0$ or $\lambda > 0$, we were able to establish both directions of this relationship with minimal requirements on K. To prove the forward direction (which holds even if $m = \lambda = 0$), we assume that K is eventually decreasing to zero and locally integrable. The complete statement appears in Proposition 3.1, which is an extension of an Abelian theorem for Fourier transforms proved by Soni and Soni [35]. To prove the backward (Tauberian) direction, we needed to further assume that K is always decreasing and placed a restriction on the $\omega \to \infty$ tail for $\mathcal{K}_{\cos}$ ($\mathcal{K}_{\sin}$). See Proposition 3.3.

Step 2: *Relating $\mathcal{K}_{\cos}$ and $\mathcal{K}_{\sin}$ to the spectral density near zero*:

$$\big[\mathcal{K}_{\cos}(\omega), \mathcal{K}_{\sin}(\omega) \sim \omega^{\alpha-1} \text{ as } \omega \to 0\big] \Longrightarrow \big[\widehat{r}(\omega) \sim \omega^{1-\alpha} \text{ as } \omega \to 0\big].$$

This critical implication in the forward direction follows from a simple analysis (Theorem 6.2), but we were not able to establish whether the reverse direction holds in general. To see why, recall the general form for the spectral density from (66):

$$\widehat{r}(\omega) = \frac{\lambda + \beta\mathcal{K}_{\cos}(\omega)}{\pi\big(|\lambda + \beta\mathcal{K}_{\cos}(\omega)|^2 + |m\omega - \mathcal{K}_{\sin}(\omega)|^2\big)}.$$

If we are allowed to assume that there exists some $\nu \in (0,1)$ such that $\mathcal{K}_{\cos}(\omega), \mathcal{K}_{\sin}(\omega) \sim \omega^{-\nu}$ near zero, then it follows that $\nu = 1-\alpha$. But it is not clear how to show that power-law behavior of the spectral density implies limiting power-law behavior in $\mathcal{K}_{\cos}$ and $\mathcal{K}_{\sin}$.

Step 3: *Relating the spectral density near zero to the large-time MSD behavior*:

$$\big[\widehat{r}(\omega) \sim \omega^{1-\alpha} \text{ as } \omega \to 0\big] \Longrightarrow \big[\mathbb{E}\left[X^2(t)\right] \sim t^{\alpha} \text{ as } t \to \infty\big].$$

This forward direction, presented in Theorem 6.2, is a substantial sharpening of the result presented in [2]. To prove the reverse direction, we believe it is necessary to better understand the following "inverse integral transform" form of the MSD (93):

$$\mathbb{E}\left[X^2(t)\right] = 2\int_{\mathbb{R}} \frac{1-\cos(t\omega)}{\omega^2}\widehat{r}(\omega)\mathrm{d}\omega.$$

Using the labels Abelian and Tauberian, one could say we have established the Tauberian side of the relationship between the MSD at large t and its transformed version, $\widehat{r}$ near zero. The Abelian side will require more work.

Finally, we note that our work does not include the critical case when $K(t) \sim t^{-1}$ as $t \to \infty$. It is not clear how the MSD of the associated GLE should behave.

On one hand, the meta-theorem implies that the MSD should scale like t, therefore being asymptotically diffusive. However, our methods require a memory kernel to be integrable for the GLE to be asymptotically diffusive. We believe that handling this case will require new insight.

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REFERENCES

[1] H. Cramér and M. R. Leadbetter, *Stationary and Related Stochastic Processes: Sample Function Properties and Their Applications*, John Wiley & Sons, New York, 1967.
[2] G. Didier, S. A. McKinley, D. B. Hill, and J. Fricks, *Statistical challenges in microrheology*, J. Time Series Anal., 33 (2012), pp. 724–743.
[3] D. Ernst, M. Hellmann, J. Köhler, and M. Weiss, *Fractional Brownian motion in crowded fluids*, Soft Matter, 8 (2012), pp. 4886–4889.
[4] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 2, John Wiley & Sons, 1966.
[5] J. Fricks, L. Yao, T. C. Elston, and M. G. Forest, *Time-domain methods for diffusive transport in soft matter*, SIAM J. Appl. Math., 69 (2009), pp. 1277–1308, https://doi.org/10.1137/070695186.
[6] I. Golding and E. C. Cox, *Physical nature of bacterial cytoplasm*, Phys. Rev. Lett., 96 (2006), 098102.
[7] I. Goychuk, *Viscoelastic subdiffusion: From anomalous to normal*, Phys. Rev. E, 80 (2009), 046125.
[8] I. Goychuk, *Viscoelastic subdiffusion: Generalized Langevin equation approach*, Adv. Chem. Phys., 150 (2012), pp. 187–253.
[9] M. Hairer, *An Introduction to Stochastic PDEs*, preprint, https://arxiv.org/abs/0907.4178, 2009.
[10] E. J. Hall, M. A. Katsoulakis, and L. Rey-Bellet, *Uncertainty quantification for generalized Langevin dynamics*, J. Chem. Phys., 145 (2016), 224108.
[11] D. B. Hill, P. A. Vasquez, J. Mellnik, S. A. McKinley, A. Vose, F. Mu, A. G. Henderson, S. H. Donaldson, N. E. Alexis, R. C. Boucher, and M. G. Forest, *A biophysical basis for mucus solids concentration as a candidate biomarker for airways disease*, PloS One, 9 (2014), e87681.
[12] C. Hohenegger, *On equipartition of energy and integrals of Generalized Langevin Equations with generalized Rouse kernel*, Commun. Math. Sci., 15 (2017), pp. 539–554.
[13] C. Hohenegger and S. A. McKinley, *Reconstructing complex fluid properties from the behavior of fluctuating immersed particles*, SIAM J. Appl. Math., 78 (2018), pp. 2200–2226, https://doi.org/10.1137/17M1131660.
[14] C. Hohenegger and S. A. McKinley, *Fluid–particle dynamics for passive tracers advected by a thermally fluctuating viscoelastic medium*, J. Comput. Phys., 340 (2017), pp. 688–711.
[15] T. Indei, J. D. Schieber, A. Córdoba, and E. Pilyugina, *Treating inertia in passive microbead rheology*, Phys. Rev. E, 85 (2012), 021504.
[16] A. Inoue, *On Abel-Tauber theorems for Fourier cosine transforms*, J. Math. Anal. Appl., 196 (1995), pp. 764–776.
[17] K. Itô, *Stationary Random Distributions*, Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math., 28 (1954), pp. 209–223.
[18] G. R. Kneller, *Generalized Kubo relations and conditions for anomalous diffusion: Physical insights from a mathematical theorem*, J. Chem. Phys., 134 (2011), 224106.
[19] S. Kou and X. S. Xie, *Generalized Langevin equation with fractional Gaussian noise: Subdiffusion within a single protein molecule*, Phys. Rev. Lett., 93 (2004), 180603.
[20] S. C. Kou, *Stochastic modeling in nanoscale biophysics: Subdiffusion within proteins*, Ann. Appl. Stat., 2 (2008), pp. 501–535.
[21] R. Kubo, *The fluctuation-dissipation theorem*, Rep. Progr. Phys., 29 (1966), pp. 255–284.
[22] R. Kupferman, *Fractional kinetics in Kac–Zwanzig heat bath models*, J. Stat. Phys., 114 (2004), pp. 291–326.
[23] M. Lysy, N. S. Pillai, D. B. Hill, M. G. Forest, J. W. Mellnik, P. A. Vasquez, and S. A. McKinley, *Model comparison and assessment for single particle tracking in biological fluids*, J. Amer. Statist. Assoc., 111 (2016), pp. 1413–1426.

[24] M. Magdziarz, A. Weron, K. Burnecki, and J. Klafter, *Fractional Brownian motion versus the continuous-time random walk: A simple test for subdiffusive dynamics*, Phys. Rev. Lett., 103 (2009), 180602.
[25] T. G. Mason, *Estimating the viscoelastic moduli of complex fluids using the generalized Stokes–Einstein equation*, Rheol. Acta, 39 (2000), pp. 371–378.
[26] T. G. Mason and D. Weitz, *Optical measurements of frequency-dependent linear viscoelastic moduli of complex fluids*, Phys. Rev. Lett., 74 (1995), pp. 1250–1253.
[27] S. A. McKinley, L. Yao, and M. G. Forest, *Transient anomalous diffusion of tracer particles in soft matter*, J. Rheol., 53 (2009), pp. 1487–1506.
[28] R. Morgado, F. A. Oliveira, G. G. Batrouni, and A. Hansen, *Relation between anomalous and normal diffusion in systems with memory*, Phys. Rev. Lett., 89 (2002), 100601.
[29] H. Mori, *Transport, collective motion, and Brownian motion*, Progr. Theoret. Phys., 33 (1965), pp. 423–455.
[30] G. A. Pavliotis, *Stochastic Processes and Applications*, Springer, 2014.
[31] F. Reif, *Fundamentals of Statistical and Thermal Physics*, Waveland Press, 1965.
[32] W. Rudin, *Principles of Mathematical Analysis*, 2nd ed., McGraw-Hill, New York, 1964.
[33] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions: Theory and Applications*, 2nd ed., De Gruyter Stud. Math. 37, Walter de Gruyter, Berlin, 2012.
[34] K. Soni and R. Soni, *The Parseval relation and monotone functions*, J. Math. Anal. Appl., 48 (1974), pp. 633–645.
[35] K. Soni and R. Soni, *Slowly varying functions and asymptotic behavior of a class of integral transforms*. I, J. Math. Anal. Appl., 49 (1975), pp. 166–179.
[36] K. Soni and R. Soni, *Slowly varying functions and asymptotic behavior of a class of integral transforms*. II, J. Math. Anal. Appl., 49 (1975), pp. 477–495.
[37] T. M. Squires and T. G. Mason, *Fluid mechanics of microrheology*, Annu. Rev. Fluid Mech., 42 (2010), pp. 413–438.
[38] E. O. Tuck, *On positivity of Fourier transforms*, Bull. Austral. Math. Soc., 74 (2006), pp. 133–138.